Using Discrete Mathematics in the K-12 Classroom to Inspire Students to Enjoy Mathematics

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Contents

Abstract

This thesis discusses three topics of discrete mathematics that can be used to teach students abstract and theoretical mathematics at an early age in conjunction with teaching them the standard curriculum. For each topic the thesis will discuss two sample problems that can be incorporated into the curriculum. In each of the sections, the thesis will include background information, a proposed lesson plan, possible extensions, and the National Council of Teachers of Mathematics standards met by the activity are discussed.

The topics covered included Cryptology, Graph Theory and Combinatorial Game Theory. The discussion of Cryptology includes two sections: Caesar ciphers and substitution ciphers. The discussion of Graph Theory also includes two sections: The Chinese Postman Problem and The Four-Color Problem. The Discussion of Combinatorial Game Theory includes three sections covering Latin squares, Tic-Tac-Toe, and the math of SET® with various restrictions.

Chapter 1

Introduction

This thesis will discuss five topics related to discrete mathematics. The first topic determines the difference between discrete mathematics and continuous mathematics by looking at how continuous mathematics prepares students to take Calculus while discrete mathematics can be taught either as a separate class or in conjunction with the current curriculum. The second topic looks at some of the reasons students should study discrete mathematics by understanding why students, after their first years of mathematics, begin to have a misconception of what mathematics truly looks like. This thesis will conclude by investigating three areas of discrete mathematics, which can be beneficial for K-12 students: using cryptology to encode and decode messages with one another, learning two topics in Graph Theory, the Chinese Postman Problem and the Four-Color Theorem, and learning the mathematics behind playing mathematical games of Tic-Tac-Toe and SET®.

1.1 Difference Between Continuous and Discrete Mathematics

I begin by considering the difference between discrete mathematics and continuous mathematics. Discrete mathematics can be thought of as discontinuous mathematics; that is it can be thought of as the opposite of continuous mathematics. John Renze and Eric W. Weisstein provide the a definition of Continuous mathematics from the website "MathWorld". Renze and Weisstein explain that continuous mathematics is the branch of mathematics that deals with "objects that vary smoothly" [34]. Calculus is the course in mathematics that for most students can be considered the culmination of continuous mathematics. Mathematics courses that are taught in K-12 are taught using continuous mathematics. The goal in teaching this form of mathematics is to have all students work toward developing their mathematical skills so they will eventually be able to take Calculus. From the beginning of their elementary school days, students are introduced to Algebra and continue to build their Algebraic mathematics skills through junior high and high school working towards the goal of taking Calculus. However, many students never plan to take Calculus and some may never reach the point where they are mathematically ready to be able to take Calculus. Furthermore, as many majors do not require Calculus, most students will have no reason to take such a course. In contrast, many of the topics covered in Discrete mathematics have direct applications to real world situations and thus have more value for students in all courses of study.

As students start on their track toward Calculus, many do not understand the purpose of continuous mathematics in general and what purpose continuous mathematics serves in their education in particular. John Dossey in his book section, "Discrete Mathematics: The Math for Our Time", addresses this issue by saying, "Continuous mathematics is well suited to situations whose main objective is the measurement of a quantity" [14]. In contrast, Dossey says discrete mathematics is fundamentally the study of "finite decision making" [13] in his book *Discrete Mathematics and the Secondary Curriculum*. Another way to look at discrete mathematics is to consider that discrete mathematics students are dealing with countable sets or finite groups of numbers, where as in continuous mathematics the students consider, work with, and study uncountable sets and are looking at gradual change over time. Dossey describes the purpose of discrete mathematics as the "focus of determining a count" [14].

Discrete mathematics problems can be broken up into three different categories: optimization, counting, and existence. In optimization, students try to find the best solution for a given problem. An example of an optimization problem is finding the quickest and/or cheapest way to get from point A to point B. In counting, students are trying to determine the number of solutions to a given problem. An example of a counting problem is to utilize an item such as a license plate. The student would be able to figure out how many different possible plates can be made given a set number of characters that are used in a certain state. The last of the three categories of discrete mathematics is existence. In an existence problem, students look to see if a route exists. For example, looking back at the travel problem, the student would first determine if a solution does exist. Then, students would go through the process of finding either the number of solutions or the optimal solution. In practice, one rarely stops at determining that a solution exists; there is almost always a further step. In a guest editorial called "Discrete Mathematics for all: Overview of Discrete Mathematics in Prekindergarten through Grade 12," Eric W. Hart states that discrete mathematics can be described as covering these topics: "vertex-edge graphs, systematics counting, iteration and recursion, matrices, voting methods, and fair division" [24]. Hart provides a more detailed and more specific description of the topics that discrete mathematics covers. Each of these areas provide K-12 students with the ability to deepen their understanding of mathematics. Discrete mathematics can also inspire struggling students to become interested in mathematics and its connections to other areas. In discrete mathematics students are able to use and apply mathematics to everyday situations. Students are able to learn about cities, find the optimal garbage pick up route, or determine what is the most efficient way to schedule meetings for the state legislature.

1.2 The Benefits of Incorporating Discrete Mathematics in the K-12 Classroom

Why should students learn discrete mathematics in their K-12 education? One of the reasons that students should be exposed to discrete mathematics early on is to get unmotivated students to enjoy and appreciate the beauty of mathematics. Dr. Margaret J. Kenney, Professor of Mathematics at Boston College, writes in the *Journal of Education* that discrete mathematics is practical and helpful in dealing with countable solutions [30]. For many students, mathematics has no direct connection to what they are learning and see in life outside the classroom. Often students ask the question, "When am I ever going to use this?" Students who eventually take Calculus discover the response to this question: "in Calculus class". Once the higher level of mathematics is taken, connection between the math they are learning and problems they encounter outside the classroom will make more sense to the students. Unfortunately, most students never make it to Calculus and never really grasp the usefulness of mathematics. If discrete mathematics is introduced to students in their elementary and secondary years, students will be able to make the direct connection between the mathematics they are learning and how it affects life around them. Through this connection, students will be able to have more success in mathematics and will be less likely to develop a hatred, dislike, or fear toward mathematics. Joseph G. Rosenstein writes in *DIMACS Series in Discrete Mathematics and Theoretical Computer Science: Discrete Mathematics in the Schools*, that "Discrete mathematics offers a new start. For the student who has been unsuccessful in mathematics, discrete mathematics offers the possibility of success" [36]. Discrete mathematics offers a number of opportunities to students who struggle with mathematics as well as students who are naturally good at mathematics. Rosenstein continues by pointing out that discrete mathematics tends to be more puzzle like problems and problems that have a direct application to real life situations [35]. Students feel more inclined to study and learn mathematics when they see a direct connection to real life by allowing students who both struggle with mathematics and those students who are naturally good at mathematics to study discrete mathematics topics. The teacher is able to bridge the gap for students who do not see any connection between mathematics and the real world, and the teacher is also able to strengthen each student's mathematical background and allow each student to become successful in mathematics. Rosenstein says it best when he writes, "One can imagine students engaged in discrete mathematics saying, "This is how I would like to spend my professional life," as well as "this is fun"" [35]. Kenney explains that discrete mathematics deals with the more day-to-day material and issues compared to continuous mathematics [30]. Since discrete mathematics can be used for example to maximize a person's success or profit or to determine what is the fastest and least expensive route to travel on a vacation, students can relate to the material better. Students who are behind mathematically or students who tend to be more mathematically challenged are able to benefit from discrete mathematics. Discrete mathematics is able to help students with lower academic scores to develop and strengthen their mathematical foundation and confidence.

Another benefit for students who take discrete mathematics at the secondary level is that discrete mathematics is separate from other mathematics classes, which allows students to be able to work discrete problems without having to have taken specific mathematics courses prior. This aspect of discrete mathematics is extremely helpful for students who do not like mathematics but is also beneficial for those students who are gifted at mathematics and tend to be bored with the traditional mathematics courses. Kenney further explains that "[d]iscrete mathematics problems and activities routinely call for group work, continuous interaction, and lively discourse before a solution can be reached" [30]. Hart agrees with Kenney, stating that he believes discrete mathematics can "provide success for students who previously may have been unsuccessful or alienated from mathematics" [24]. Hart goes further, explaining how discrete mathematics can be taught at any grade level, matching the cognitive level of the student to the level of mathematic and academic rigor. He further notes that students are able to sort different types of buttons in early grades, count different flag patterns in middle school, and plan a school dance in high school by using vertex-edge graphs and critical path technique [24]. Students who are exposed to discrete mathematics early on are able to see why certain methods of solving problems are more effective than others for given situations. According to Hart, students "develop new types of reasoning, such as combinatorial reasoning, which is used to reason how many different pizzas are possible when you choose two out of five toppings" [24]. Since our culture is becoming more technology driven, the need for students learning more mathematics dealing with technology is becoming more and more relevant, and the need for learning technological mathematics and how computers process and optimize problems is becoming more critical for students to be successful in their future jobs.

Nancy Casey and Michael Fellows further explore the reasons why discrete mathematics needs to be implemented in the early elementary years. In the article, "Implementing the Standards: Let's Focus on the First Four", Casey and Fellows write that students at the K-4 level need to be exposed to more "high-level cognitive issues and away from the traditional focus on the accumulation of low-level rote computational skills" [10]. Casey and Fellows asked students after the first year of school their thoughts about mathematics. Many students did not like mathematics and believed "it a boring and intimidating discipline devoted primarily to speedy and accurate manipulations of numbers" [10]. Casey and Fellows's research into how the students felt after their fourth year of school provided further evidence of the need for a change from the traditional memorization of mathematics facts to a high-level cognitive learning. Casey and Fellows noted that by the end of the fourth year students have typically had an abundance of the traditional experiences of school mathematics: the meaningless seat work, memorization of procedures, the stilted word problems and pointless obscure vocabulary, the anxiety of parents and teachers, and the testing that separates the winners from the losers [10]. Casey and Fellows explain that even at this age students learn once they have successfully passed one mathematical practice sheet that they are only rewarded with another practice sheet [10]. Casey and Fellows further explain that students at this age are surrounded by the "true spirit of mathematics" on the playground in which they play on every day [10]. As Casey and Fellows explain, the students' "playground culture is rich with combinatorial games, with riddles and word-play, with informal discussions of infinity, space-time, and the Lair Paradox. They are busy with topological and dynamic amusements such as tether ball, jump rope, cat's cradle and braids" [10]. Casey and Fellows believe experiences during the K-4 years are vital for developing success and love for mathematics [10]. If students lose the ability to enjoy mathematics at an early age, then the students' teachers later on may have a hard time rekindling that enjoyment. Children at the K-4 level have more curiosity and enthusiasm, which are often lacking as the students get older [10].

Another added advantage of incorporating discrete mathematics into the current curriculum is the ability for teachers to help students develop and strengthen their problem solving skills. Often after state testing, students are no longer motivated to learn new material. By allowing discrete mathematics to be taught and used in the K-12 classroom, students will no longer feel as if they are being required to learn new mathematics after state testing. Students will, however, be able to participate in open discussions utilizing their critical thinking skills to solve problems that do not necessarily come from a textbook. Rosenstein explains "[w]ith discrete mathematics, students can easily become involved in the doing of mathematics, can see themselves as 'mathematicians' rather than as followers of routine instructions" [36]. Another important aspect of learning that discrete mathematics provides to K-12 students is access to new and current mathematical topics. Rosenstein explains how discrete mathematics "consists of many topics, which lends themselves readily to approaches to learning that are recommended in the national reports: discovery learning, experimentation, problem solving, cooperative learning, and use of technology" [36]. The approaches Rosenstein writes about are all important for students not only in an increasingly technological society, but also for students who are growing up in a society where they are not being required to develop problem solving skills of their own. Also, students must learn to be able to work together and learn to get along and to work toward a common goal of solving the problem. By introducing discrete mathematics into schools' curricula via group problems and activities, students will not only develop more mathematical skills, but will also be able to grow more socially.

In each of the following chapters I will present two problems that are representative of the type of problems one might encounter in the chosen area of discrete mathematics. Each problem will include background information, a proposed lesson plan, possible extensions, and a discussion of the National Council of Teachers of Mathematics (NCTM) standards met by the activity.

Chapter 2

Applications of Cryptology in the Classroom

According to *The Columbia Electronic Encyclopedia*®, cryptology can be defined as the "science of secret writing" [45]. As technology has advanced, so has the need to keep information hidden. Early militarized civilizations used cryptology to keep information hidden from their enemies. Even in today's modern culture, people want to be able to conceal and hide important information such as Military Operations, Submarine locations, E-mail passwords, ATM numbers, Social Security Numbers, Bank Account Information, and other private information. Cryptology has been a vital tool for keeping such information hidden, especially from those who would use private information to harm other individuals. Students at a young age are able to grasp the types cryptology that early civilizations used to keep secrets. For example, students at K-12 level can learn how to use a Caesar cipher and a substitution cipher to encrypt messages and to decrypt encoded messages using these ciphers.

2.1 Caesar Ciphers

According to Simon Singh in his article "Arab Code Breakers", one of the earliest forms of hidden communication was the Caesar cipher. Julius Caesar, for whom the cipher was named [42], used the shift method to encrypt messages to his generals. The shift method shifts the starting position of the *plaintext*, or original alphabet, over a certain number of spaces to create the *ciphertext*, or encoding alphabet. Janet Beissinger and Vera Pless describe how the Caesar cipher works in their book, *The Cryptoclub*: "In a Caesar cipher, the alphabet is shifted over a certain number of places and each letter is replaced by the given corresponding shifted letter" [3]. Julius Caesar shifted the text three positions "so that plaintext A was encrypted as D, while Augustus Caesar used a shift of one position so that the plaintext A was enciphered as B" [40]. An example of a message that could have been sent using the Caesar cipher would be:

Move the troops to the North. Becomes, PRYH WKH WURRSV WR WKH QRUWK when using the Caesar Cipher by shifting the alphabet over 3 spaces so that the letter *a* starts at *d* in the *ciphertext*.

plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M
ciphertext	D	E	F	G	H	I	J	K	L	M	N	O	P
plaintext	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
ciphertext	Q	R	S	T	U	V	W	X	Y	Z	A	B	C

In a document produced by the Central Intelligence Agency (CIA), the CIA describes the Caesar cipher as "fun to use;" however, according to the CIA the cipher is not a secure way to communicate [11].

When considering different ciphers, a person must know and understand the algorithm being used to encode the message. In the Caesar cipher, the algorithm is replacing the *plaintext* letters with the shifted *ciphertext* letters. The *key*, in the Caesar cipher, is the number of places the text alphabet was moved [42]. The key in a Caesar cipher is always able to be changed and never has to remain the same each time a person wants to encode a message. Singh explains that the "key is usually selected by the sender and has to be communicated to the receiver so that the message can be deciphered" [42]. For any cipher to be useful, the receiver of the message must be able to decrypt the message. By having a key, the receiver of the encoded message is able to decipher what the original sender sent. Singh explains that code breakers are able to decode the message, if intercepted, by figuring out the key [42]. The Caesar cipher is not overly complicated. If a message was intercepted and the original message was encrypted using a Caesar cipher, the person who intercepted the message could simply use brute force and test all possibilities, since there are only 25 possible keys [42].

2.1.1 History of the Caesar Cipher

Julius Caesar is said to have described how he encoded a message to Cicero in the book, *Gallic Wars* [42]. According to Simon Singh, Caesar is said to have encrypted a message "by substituting Greek letters for Roman letters, then delivered [it] in the most dramatic way imaginable" [42]. Caesar had sent a messenger to deliver the message to Cicero; however the messenger was not able to reach Cicero's camp. The messenger instead had to tie the message to a spear and throw the spear, which got stuck in a tower and was not noticed by anyone for a few days. Once the spear was noticed, the message was taken down and taken to Cicero. According to historic records, once the message was delivered to Cicero, he "read out the vital news to the entire camp, bringing enormous joy to his troops" [42]. The Caesar cipher had become an important tool for Caesar to be used for his military operations, allowing for messages to be hidden from his already illiterate enemies. Even if messages were intercepted by a literate opponent, the enemy was unable to understand the true meaning of what was written without knowing how the messages were encrypted or what language the message was encrypted in.

2.1.2 The Caesar Cipher in the Elementary Classroom

The Caesar cipher can be a good way for students in a K-5 Classroom to be introduced to coding, modular arithmetic, patterns, and frequency. The teacher should spend one or two days on introducing this activity to the students in an elementary classroom. This lesson would mostly likely be introduced after the students have covered and mastered addition and subtraction since the Caesar cipher shifts the alphabet x number of spaces to the left or to the right of the original starting place. Lance Bryant and JoAnn Ward created a lesson plan to be used when covering the Caesar Cipher. Bryant and Ward suggest that the teacher start by spending part of the day explaining the history of the Caesar Cipher and who it was named after. The students can also spend the last part of the day creating a Caesar wheel, which is used in decoding messages. See Appendix B for a copy of the Caesar wheel. The main purpose of the lesson should be to expose the students to challenging problems and allow students to see how mathematics can be used. Bryant and Ward explain that there is little or no computational mathematics involved: "In fact, there is only one number (between -25 and 25) that is used in the cipher as the encryption key" [5]. Another important purpose of this lesson is that it allows students problem solving skills to be strengthened.

According to Byrant and Ward [5], the main objectives for this lesson is to:

1. Introduce students to an interesting area of Mathematics
2. Challenge students' conception of mathematics
3. Analyze students' of the strengths and weaknesses in using the Caesar cipher and using this analysis to compromise the system and to improve it.

The NCTM Standards, found in Appendix A, this lesson will address will be:

Pre-K-2 Standards 1, 2.a, 2.b, 2.c, 5.a, 5.b, 5.c, 6, 7, 8, 9, 10

Grades 3-5 Standards 1, 2.a, 2.b, 2.c, 5, 6, 7, 8, 9, 10

Consider for example Pre-K Standard 2.2a: "Understand patterns, and functions." When studying a Caesar cipher, students will be expected to figure out the patterns that *ciphertext* contains. By finding and understanding the patterns involved within the text, students will be able to use Algebra to determine how many places to shift the *plaintext*, or, if they are working as decoders how many places the *ciphertext* was shifted over. Another example would occur in Grades 3-5 Standard 6.b: "Solve problems that arise in mathematics and in other contexts." When learning the Casar cipher, students are learning how to use basic mathematics in settings outside the classroom. Students are able to see how early military leaders were able to keep messages hidden from their enemies.

The Lesson: Once students are familiar with the history of the Caesar cipher and have made their own Caesar wheel, the teacher should have the students work together in groups breaking a simple coded message, one that is only shifted +3 so that A=D, B=E, C=F, D=G, E=H, ..., Y=B, Z=C. On the board, the teacher should write the following *ciphertext*:

ZULWLQJ KLGGHQ PHVVDJHV LV IXQ

After the students have spent some time deciphering the message, place the correct deciphered message on the board, (writing hidden messages is fun), allowing the students to make sure they got the correct answer. Doug Schmid suggested when doing this lesson plan, "Point out that this coding system . . . [is] easy to break; even without knowing the coding system, expert code breakers are able to decipher them in minutes or even seconds" [38].

Allow students to spend time in class encoding and decoding messages with one another using the Caesar wheel. After the students have completed a few problems, they should be able to see if the codes are easy to break and be able to explain why more complex codes are needed.

The teacher should consider having the more gifted or mathematically advanced students doing the following: Consider giving the advanced students a coded messaged without including the key number. By not sharing with the students how many letters the text has been shifted, students must try and figure out the *plaintext* by looking for patterns. As a hint, the teacher can explain that certain letters such as e, a, t, and n are the most frequent occurring letters in the English language [5]. Having students look at which letters occur the most in an encrypted message, students can then use frequency analysis to break codes that are not as obvious. Frequency analysis will be covered in further detail in next the section on substitution ciphers.

By incorporating this lesson within the curriculum, students are able to make connections with other fields outside of mathematics. Bryant and Wade further explain that this lesson helps students who are more interested in the language arts or who enjoy studying languages [5].

The evaluation for this lesson will be in the form of worksheets such as those provided in Appendix B. Now that the students have spent time learning how to encode and decode the Caesar cipher, the teacher should hand out the worksheets and have the students either work by themselves or allow them to work in pairs. Provide students with plenty of time to work through the problems. Before the day ends, bring the students back together and go over the answers to the worksheet.

One possible extension the teacher may also want to show students would be to use numbers to encrypt messages as well.

plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M
ciphertext	0	1	2	3	4	5	6	7	8	9	10	11	12
plaintext	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
ciphertext	13	14	15	16	17	18	19	20	21	22	23	24	25

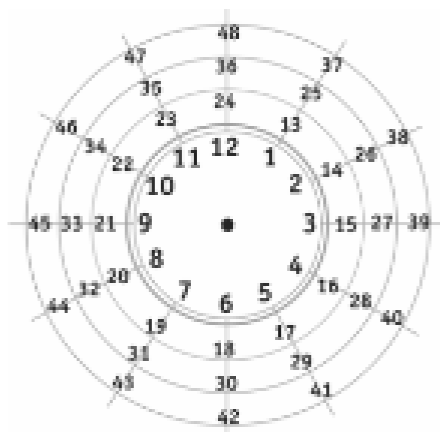
Here is one possible exercise using a numerical cipher that an elementary teacher can use that Bessinger and Pless suggest [3]:

1. Use the number method to encrypt your teacher's name. Compare you answer with the rest of the class.

2. Use the number method to encrypt your name. Put your encrypted name in a "hat" that your teacher provides.

3. Pass the hat around and pull a name from it. Decrypt the name and return it to its owner.

Bryant and Wade also point out using the Caesar cipher in the classroom can be used as a single lesson or one that can introduce other topics such as: frequency, modular arithmetic, and bar graphs [5]. One particularly nice extension is using methods of Caesar ciphers, and the Caesar wheel, to explore modular arithmetic. Since students are accustomed to working with modular arithmetic when using standard time, students already have a basic understanding that when they go past 12, they start back over at 1 and go back through to 12. This is fundamental idea behind what we call modular arithmetic. When extending this lesson, students have the ability to work with larger numbers. Students could be given the following modular clock that was provided by Beissinger and Pless when working with modular arithmetic [4].



Using this numeric modular cipher students now have the ability to see how Caesar cipher and the numeric cipher tie into one another. Students can see that when using numbers larger than 12, they can reduce the numbers to a number between 0 and 11. The teacher now can ask questions using time and months. For example, the teacher could ask the students to name the month that occurs 36 months from March? Or what month was it 23 months ago? What time will be 47 hours from 2:00 A.M.? The teacher also has the ability extend modular arithmetic even further by changing the modulus that the students had been working in. For example, the teacher could change the mod from 12 to 7, so that the teacher can ask questions about days of the week. By extending this lesson, the students could spend time developing cipher wheels for different moduli. If the teacher desires, further connections can also be made to long division and remainders, mixed numbers and improper fractions, as well as other topics from the typical curriculum.

2.2 The Substitution Cipher

Another type of cipher appropriate for introduction in a middle school classroom is the substitution cipher. Beissinger and Pless provide a description of a substitution cipher, which is a cipher where "each letter of the alphabet substitutes for another letter" [3]. Beissinger and Pless explain that the Caesar cipher is an example of a substitution cipher; however, since the Caesar cipher is only a shift substitution cipher the Caesar cipher is an easy cipher to solve [3]. Beissinger and Pless explain that "[e]ven though most substitution ciphers are not as easy to crack as Caesar ciphers, cryptographers can break them fairly easily" [3]. Since the Caesar cipher is only shifting the letters of the alphabet a certain number of spaces, a more complex and harder cipher to crack is "the general substitution cipher, which allows the cipher alphabet to be any rearrangement of the alphabet" [42]. Singh explains that "there are roughly 400,000,000,000,000,000,000,000,000 possible keys, because this is the number of ways to rearrange the alphabet" [42]. Even though this is a larger number and checking each possible key would take a long time, Singh says "a large number of keys is not the sole requirement for a secure cipher, but it certainly makes the codebreaker's job harder" [42]. Substitution ciphers do not have to be limited to just letters of the alphabet. People who are wanting to use a substitution cipher could use symbols to stand for letters [3]. The author of *Sherlock Holmes*, Sir Arthur Conan Doyle, used a substitution cipher in one of his stories, "The Adventure of the Dancing Man." In this story, Doyle used stick figures to represent letters. Doyle loved cryptography and used ciphers in some of his writings [3]. When trying to solve codes where substitution determines how the message was encoded, using letter frequencies to find encoded letters can be helpful for the person decoding the message. As Singh explains, one of the ways to decipher the coded message would be to guess or know part of the message, "which is known as a crib" [42]. By knowing the crib, the cryptographer is able to use deductive reasoning to decode the rest of the message. The example that Singh gives is “the codebreaker might know that a *ciphertext* that begins ‘XBKJ . . . ’ is a letter, so XBKJ might stand for ‘dear . . . ’, which means that the true values for four letters have been established which in turn might be helpful in deciphering the rest of the message” [42]. The cryptographer can then search the rest of the message and replace the *ciphertext* letters with the *plaintext* letters that XKBJ represent, helping the cryptographer to decoded and piece together more of the coded message.

2.2.1 History of the Substitution Cipher

Al-Kindi, ninth-century Arab philosopher, wrote the paper, “A Manuscript on Deciphering Cryptographic Messages,” which describes a method now referred to as “frequency analysis” [42]. Simon Singh explains on his website that Al-Kindi was in charge of the “House of Wisdom, a research institute and library, based in Baghdad” [42]. Al-Kindi was in charge of translating and trying to store as much information from around the world as he could. Some of the documents that Al-Kindi received were encrypted, and Al-Kindi did not want any knowledge to be left hidden. Since these messages were encrypted, Al-Kindi developed a passion for discovering the true meaning of the encrypted texts [42].

Al-Kindi was the first to break the general substitution cipher. As Al-Kindi was studying the coded messages, he began to notice the frequency of certain letters. Al-Kindi began to see patterns when he began to substitute letters, frequently used in Arabic, into the coded message so that the true word became apparent [41]. Beissinger and Pless provide a chart for letters in the English language and how often these letters occur. The the most frequent letter that occurs in the *ciphertext* might represent the most frequent occurring letter in the English language which is the letter “e.” In a study of 40,000 words, 12.74% of the letters used were “e.” In that same study 9.1% of the letters were “t,” 8.2% were “a.” 7.5% were “o,” and 6.7% were “i” [3]. Beissinger and Pless also list the least occurring letters in the English language, which are “q,” “x,” and “z” all occurring with a frequency of 0.1% in the 40,000 English words studied [3]. By understanding this type of frequency of letters, Al-Kindi was able to figure out what letters to substitute in and where, thus allowing the message to be deciphered. Singh writes "Al-Kindi's invention was based on new mathematical techniques that were being developed by the Arabs and on a deeper understanding of the structure of the language and of writing, which was driven by a desire to gain a deeper insight into the Koran" [42]. Al-Kindi's "A Manuscript on Deciphering Cryptographic Messages" not only developed frequency analysis, but is also the first known paper to discuss statistics [42]. Using frequencies to solve substitution ciphers made its way to Europe where the method was used to break the codes of enemies. Thomas Phelippes was able to use frequencies to prevent the assassination of Elizabeth I and replace her with Mary Queen of Scots in 1587 [42]. Singh continues by explaining that Mary Queen of Scots chose to use a weak cipher during the 16th century, "long after codebreakers had mastered frequency analysis" [43]. Mary Queen of Scots and Anthony Babington exchanged messages back and forth plotting to assassinate Queen Elizabeth I, which is called the "Babington Plot" [43]. Mary tried to conceal her plan by using a cipher to keep her plan secret; however, as Singh states: "Mary's messages were captured by Elizabeth's spies and they were cracked by her chief codebreaker. Mary was immediately arrested, put on trial and the deciphered messages were used as evidence of her treachery. She was found guilty and was executed in 1587 . . . all because her cipher was cracked" [43].

2.2.2 The Substitution Cipher in the Junior High School Classroom

The substitution cipher can be a good tool for building on the foundational understanding that students developed in our exploration of Caesar ciphers in elementary school. Students in a 6-8 Classroom will further develop an understanding of coding, modular arithmetic, patterns, and frequency. The teacher can spend one or two days on introducing this activity to students in a junior high classroom. This lesson could be introduced after the students have covered and mastered inverses, data analysis, and graphs. However, a substitution cipher can also be used to introduce these and other topics as well as to illustrate the utility of cryptography in these and other areas of mathematics.

The objectives for this lesson will be for students to:

1. Encode using a substitution cipher and decode using frequency analysis.
2. Be exposed to challenging mathematics.
3. Find connections with topics outside of mathematics
4. Find the strengths and weaknesses of the substitution cipher and how to make the cipher more secure.

The NCTM Standards which this lesson will cover are:

Grades 6-8 Standards 1, 2, 5, 6, 7, 8, 9, 10

Here, for example, are just two ways the substitution cipher meets the NCTM Standards listed. Consider Grades 6-8 Standard 5.b: “Select and use appropriate statistical methods to analyze data.” When deciphering a substitution cipher, students must try and find the letter frequencies that occur within the *ciphertext*. Furthermore, by finding the letter frequencies, students are able to “[a]pply and adapt a variety of appropriate strategies to solve problems,” which is Standard 6.c.

The Lesson: The lesson should begin by explaining the history of the substitution cipher and how Al-Kindi was able to develop the study of frequency analysis, which he used to solve decoded texts. The teacher may want to spend a day on the Caesar cipher or a few minutes reminding students how to encrypt and decrypt the following problems:

1. Caesar cipher with shift of +5
   ltytufwnx
2. Caesar cipher with shift of $+21 = -5$
   adiyevhznwjiy
3. Caesar cipher with shift of $+24 = -2$
   ncwrmlkyllgle

Having reviewed Caesar ciphers, the class should now consider a more general substitution cipher in which the letters have not been shifted, but rather have been randomized.

Here is an example of a substitution cipher that is not a shift cipher. Have the students decode the following message using the following key.

plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M
ciphertext	I	D	J	P	B	O	U	K	Z	F	N	T	A

plaintext	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
ciphertext	Y	G	Q	V	X	C	L	R	H	E	S	M	W

LKB LXRB CZUY GO ZYLBTTZUBYJB ZC YGL NYGETBPUB DRL ZAIUZYILZGY.

Which decodes into *THE TRUE SIGN OF INTELLIGENCE IS NOT KNOWLEDGE BUT IMAGINATION.– Albert Einstein*

Have the students encode the following message using the above key.

Better to remain silent and be thought a fool than to speak out and remove all doubt.

–Abraham Lincoln

Which encodes in:

DBLLBX LG XBAIZY CZTBYL IYP DB LKGRUKL I OGGT LKIY LG CQBIN GRL IYP XBAGHB ITT PGRDL.

Once students have spent some time encoding and decoding using the Caesar cipher and the Substitution cipher with known key the next step is to try and break a code with unknown key. This is the process that a "spy" would have to complete in order to decode an intercepted message. By moving onto this harder problem, students will begin to experience the job of a cryptologist as well as to see a connection between English and Mathematics. The teacher should begin by reviewing the common occurrence of letters in the English language. Discuss how the frequency with which letters commonly occur in the English language, frequency analysis, can be useful to a cryptologist. Schmid explains frequency analysis as "the process by which the frequency of a letter in an encoded messages is compared with the frequency of letters in English words. For instance, the letter E occurs most often in English words, so if the letter W occurs most often in ciphertext, then it is likely that E has been replaced by W" [38]. Schmid provides an example where frequency analysis is beneficial in the following problem

TFNRIUJ UZV DREP KZDVJ SVWFIV KYVZI UVRKYJ;
KYV MRCZREK EVMVI KRJKV FW UVRKY SLK FETV.

"The decoded message, taken from William Shakespeares play *Julius Caesar*, is 'Cowards die many times before their deaths; the valiant never taste of death but once"' [38].

Showing students a letter frequency chart, the students should able to start to try and figure out what letters to try and use.

Schmid explains students should begin to see that

- The letter V occurs 13 times
- The letter K occurs 9 times
- The letter R occurs 7 times

Since E is the most frequently occurring letter in the English language, then students should naturally try and substitute V with E and then try using the next most frequent occurring letter in the English language T by substituting T in for K. By this time students should notice the word after the semicolon is T_E and the missing letter is most likely a H. Thus, one of the words is THE. At this point the teacher can have the students finish the rest of the problem at home. Some students might notice, once the word THE has been deciphered, that a Caesar shift of 17 was used to encode the message.

The second day of the lesson should begin with students going over the answer and the problems that they might have had. It should be at this point in time that the teacher points out the message was encoded with a Caesar shift of 17.

Once students are comfortable with using a frequency table, the teacher should ask, "How could we make this more difficult to crack?" By asking the students, "What would happen if each letter of the alphabet was assigned a completely random letter and the *ciphertext* was no longer shifted over x amount of spaces," the teacher will help students realize that this method will indeed make the message more difficult to crack. However, students should remember that frequency analysis will still help decipher hidden messages.

Beissinger and Pless provide a helpful list when using letter frequencies to decrypt codes:

- Match the most common letters first-you'll make faster progress
- Use relative frequencies to help, but don't expect them to match exactly
- One you know some letters of a word, try to guess others until you have a word that makes sense.
- Look for familiar short words. One-letter words are usually **a** or **I**. Two and three letter words such as **in**, **of**, **at**, **and** and **the** are helpful.
- Let punctuation help–for example, what letters can follow an apostrophe?
- Look for pairs of letters, called **digraphs**, that often occur together. Among the digraphs most common in English are TH, HE, IN, ER, ED, AN, ND, AR, RE, and EN. Common groups of three letters, called **trigraphs**, are THE, AND, ING, HER, THA, ERE, GHT, and DTH.

The teacher should place this problem on a board or overhead and allow students to begin to think of how to solve the problem.

ILN'X WQ HDKHUI XL YQQ BAHX ELO YQQ.

Students should begin to count and find which letters occur the most. By using letter frequency students should see that Q occurs the most and is probably an E. By substituting E in for QQ, they should now have the last word that ends in EE. By thinking through some of the possible words that end in EE, students could possible assume that Y in the *ciphertext* could be an L,B, S, or an F in the *plaintext*. Students can take an educated guess and decide to use the letter S to substitute in for Y. Providing students with two of the encoded words as the word "SEE," the students now have a word, they should be able to reason that since there is a contraction at the beginning of the sentence the last two will probably be an N'T. Using this, students begin to see words emerging. The first word is **DON'T**; the second word becomes **BE**; and the fifth and very last words are **SEE**. By knowing what L in the *ciphertext* translates to, the decoder can see that the seventh word is **YOU**. By this time students should have majority of the message decoded leaving.

ILN'X WQ HDKHUI XL YQQ BAHX ELO YQQ.
DON'T BE A__ __A__D TO SEE __ __AT YOU SEE.

Where the missing letters are D,K,U,B,and A. By using deductive reasoning, students should be able to solve the code to become *"DON'T BE AFRAID TO SEE WHAT YOU SEE."* –Ronald Reagan

Students should be ready for working a worksheet covering the substitution cipher and frequency analysis located in Appendix B.

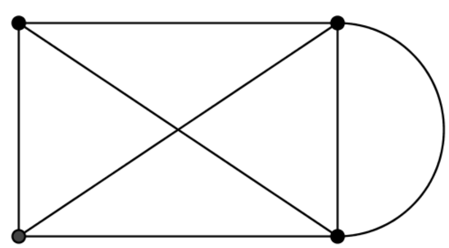
The teacher should allow students to work on the worksheet alone or in pairs. The teacher should also provide plenty of time for students to ask questions and receive hints if needed when using the substitution cipher. The teacher can either bring the class back together at the end to go over the answers or have the assignment due the following day in class. Having the assignment due the next time the class meets would allow the students to spend time using their problem solving skills to think through the assignment. The teacher needs to allow for students to ask questions when the substitution cipher is first introduced.

The teacher can also extend the lesson to teaching the students about keyword ciphers, Vigenere cipher, modular arithmetic, multiplicative and affine ciphers, and modern cryptology. The book *The Cryptoclub: Using Mathematics to Make and Break Secret Codes* by Janet Beissinger and Vera Pless provides worksheets and examples for each of the problems.

Chapter 3

Applications of Graph Theory in the Classroom

According to Eric W. Weisstein, Graph Theory can be defined as "[t]he mathematical study of properties of the formal mathematical structures called graphs" [47]. Wayne Copes, Clifford Sloyer, Robert Stark and William Sacco, in their book *Graph Theory: Euler's Rich Legacy*, define a graph as a "set of points or vertices, and a set of lines called edges which connect pairs of vertices" [46]. The following is an example of a graph in discrete mathematics.



Graph Theory has become a very useful tool for the computer and technological age of mathematics. In their paper "Euler, Mei-Ko Kwan, Königsberg, and a Chinese Postman," Martin Grötschel and Ya-

Xiang Yuan write, “[G]raph theory, a new branch of mathematics, that is today permeating almost every other science, is employed even in daily life, has become a powerful modeling language and a tool that is of particular importance in discrete mathematics and optimization” [20]. However, even though the usefulness of Graph Theory can be seen in modern mathematics, Graph Theory first started in the 1800s.

According to Copes, Sloyer, Stark, and Sacco [46], Graph Theory first came into existence in the eighteenth century as a method of solving the Bridges of Königsberg problem. Königsberg was a town in Eastern Prussia, now Kaliningrad Russia, through which the river Pregel flowed. Pregel surrounded an island called Kneiphof. The city had seven bridges that crossed the river allowing citizens of Königsberg to travel to other parts of the city easily. Some of the people of Königsberg began asking if there was a way for a person to be able to walk around the town and cross each of the seven bridges exactly once. For years, people of Königsberg tried to cross each bridge only once but failed. It was not until 1736 that Leonhard Euler, a famous mathematician, wrote a paper proving that it could not be accomplished. Euler not only proved that the Königsberg bridge problem was impossible, but he also went further to describe all the configurations for which it would be possible [20]. Other mathematicians would have only proved that the Königsberg bridge problem could not be done and stopped there, but Euler was able to distinguish himself by describing all possible configurations for which the Königsberg bridge problem would be possible.

Euler showed how to solve problems similar to the Königsberg bridge problem and not just the bridge problem itself [32]. The paper, “Solutio Problematis ad Geometriam Situs Pertinentis”, written by Euler in 1736, created the branch of mathematics now known as Graph Theory. Eric Hart, in the section “Discrete Mathematics an Exciting and Necessary Addition to the Secondary Curriculum” of the book *Discrete Mathematics Across the Curriculum K-12*, states that Graph Theory is also referred to as the “mathematics of dots and lines” [23]. Viewed in this way, Graph Theory is an accessible topic in mathematics for students to pick up. Robert Holliday writes in his section “Graph Theory in the High School Curriculum” of the same book, that many students are able to follow and understand the process and definitions that are associated with Graph Theory [26]. When introducing Graph Theory, the teacher is able to provide students with visual examples of many of the definitions, which allows students to make connections with the terms and their meanings. For example, a student does not need to understand formal definitions of words like vertex, edge, incidence, etc., but will be able to understand these concepts by looking at examples of graphs. In many math classes, students are given a definition by the teacher and are asked to memorize it, in contrast, in Graph Theory, a teacher can allow students to write a detailed definition in their own words from a picture that was drawn [26]. By allowing students to become active participants in learning how objects in Graph Theory are defined and drawn, students are able to make connections with the terms and the definitions that are associated with them. This active participation allows students to discover and learn mathematics on their own. We will consider two example problems from Graph Theory and how they can be implemented in a K-12 classroom: the Chinese Postman Problem and the Four Color Problem.

3.1 The Chinese Postman Problem

Grotschel explains that the Chinese Postman Problem can be stated as follows:

A postman has to deliver letters to a given neighborhood and back to the post-office. How can he design his whole route so that he walks the shortest distance [20]? That is, he wants to minimize the distance of the roads he walks over twice. This problem is a good one for students to consider as they can understand and relate to the Graph Theory.

3.1.1 History of the Chinese Postman Problem

The Chinese Postman Problem was posed by the 26 year old Chinese lecturer, Mei-Ko Kwan, from the Shandong Normal University in the 1960s [20]. During the late 1950s and the early 1960s, China was changing from an agrarian society to a “modern communist society” [20]. Many Chinese mathematicians were focusing on real world applications and looking at how to apply mathematics to their evolving society. Mei-Ko Kwan was one of the Chinese mathematicians who focused on transportation and optimization problems. Mei-Ko Kwan published his "Programming Method" paper in 1960, in which he solved the problem by using odd and even points. This paper was later translated into English and caught the attention of Jack Edmonds who was the first mathematician to give the problem the name, "the Chinese Postman Problem" [20].

The Chinese Postman Problem has a variety of applications. For example, it can be used by companies wanting to maximize efficiency or minimize costs. Guozhen Tan, Jinghoa Sun, and Guangjian Hou wrote a paper entitled, "The time-dependent rural postman problem: polyhedral results," where they state some applications for the Chinese Postman Problem: "Applications include mail delivery, garbage collection, snow removal, school bus transportation and very large integration circuit design and optimization in software testing" [21]. Tan, Sun and Hou went on to say that the scientific community has been able to use various different forms of the problem since the Chinese Postman Problem has many real world applications [21]. Mony Sedlak wrote an article in *Public Works Magazine*, where he added a few more areas where the Chinese Postman Problem can be used by cities. Sedlak listed buses, meter readings, street sweeping and refuse collections as other areas where the Chinese Postman Problem has become useful [39]. The real world application of Chinese postman theory is especially beneficial for cities that are on a tight budget and are looking to cut costs. Some major cities have been able to apply the Chinese Postman Problem when deciding which roads during the winter months to plow. Marcus Woo wrote "The Mathematics Behind Getting All That Damned Snow off your Street" in the magazine, *Wired*. Here, Woo explained how the theory works by stating,

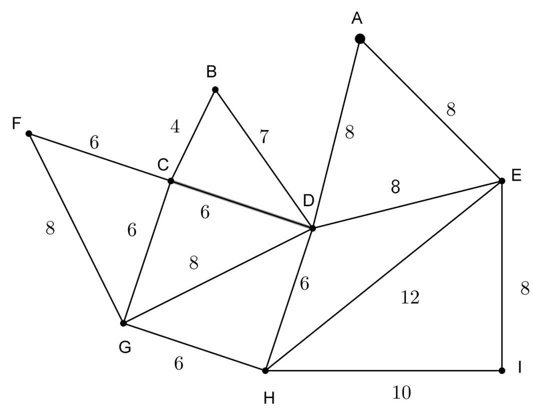
> it's about finding the most efficient routes between intersections with an odd number of streets. As Euler showed, those are the only places that force you to back track. So to find the best overall route, you have to identify all the odd intersections and all the possible ways you can travel between them, and then find the most efficient paths. Combining those with the routes covering all the even intersections gives you the most efficient route [49].

Woo continues by explaining that the Chinese Postman Problem only applies in simple situations, "for example, when you have exactly one plow. Most cites have dozens, even hundreds, each of them responsible for specific areas" [49]. Woo also mentions that "[h]ighways and expressways have priority over residential streets. Plows deploy from depots throughout the city. Cities want to minimize overtime, or ensure each driver works a similar number of hours" [49]. If a city has thousands of streets, this becomes more than a mathematical problem. Often in real world applications, there is not a perfectly optimal mathematical solution that works. However in practice, as Woo says, "You don't need the perfect solution-just one that gets the job done" [49]. Many larger cities use the computer program ArcGIS, which provides the most efficient and cost effective roads to plow for cities. Woo explains that by using ArcGIS Toronto Canada has been able to cut the cost to run all the vehicles and the amount of salt needed by 30 percent [49].

Students at the K-12 level should be able to work the Chinese Postman Problem when they are given a map of a small city or town and asked to determine the most efficient way to plow a city. By being given real world problems, students who are generally uninterested in the traditional mathematics curriculum are now able to get their hands dirty and to take part in real world examples of mathematics. Students are able to develop problem solving skills and to build confidence in themselves, which are needed to solve other mathematical problems that generally arise in the typical continuous mathematics course sequence. Susan Picker wrote in, *Discrete Mathematics in the Schools*, that she was able to see a change in students' attitudes and beliefs after students took a discrete mathematics course. Many of the students, after taking the discrete mathematics course, went back into the normal course sequence and took an algebra class, and for most of those students academic achievement was seen in their algebra class [33]. One of the major benefits that discrete mathematics offers for students at the K-12 level, is the ability to develop the students' problem solving and critical thinking skills. Students are able to develop these problem solving skills by looking at problems such as the Chinese Postman Problem in a classroom setting.

3.1.2 The Chinese Postman Problem in the High School Classroom

The Chinese Postman Problem is a useful theory to be taught in the High School mathematics classroom. Students who have studied geometry or are currently enrolled in geometry are able to understand the connections between geometry and Discrete mathematics. The following is an example of a route a postman must travel on any given day.



The postman must start at point A and end back at point A. The postman must travel each of the 14 streets. The numbers on the edges represent the distances in hundreds of steps the postman takes to cover each street. The goal is to find the path that covers every edge using the minimum amount of distance [37]. We will return to solving this example problem later.

The main objectives for this lesson will be:

1. Understand how the Chinese Postman Problem works
2. Apply an algorithm to solve the problem
3. Look for applications for the Chinese Postman Problem outside the mathematical classroom

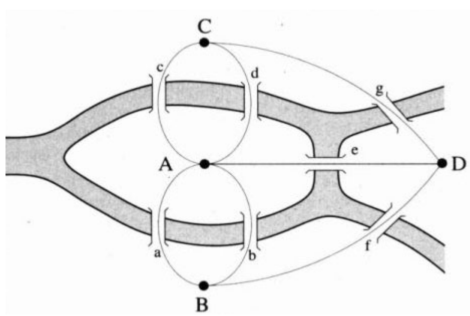
The NCTM Standards which the assignment will cover are:

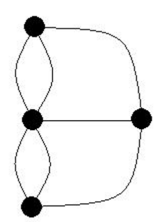
Grades 9-12 Standards 1, 2, 3, 6, 7, 8, 9, 10

Consider for example Standard 3.d: "Use visualization, spatial reasoning, and geometric modeling to solve problems." Here students will be able to look at a map of a given city and will be able to calculate the quickest and most cost effective route for the city either to clean the streets, pick up trash, or clear the streets of snow.

Also consider Standard 8.b: "Communicate their mathematical thinking coherently and clearly to peers, teachers, and others." Here students will be able to explain why their solution to the problem is the optimal solution. The Chinese Postman Problem allows students to have students develop their own arguments and provides students with the ability to justify their reasoning to other students, peers, and teachers.

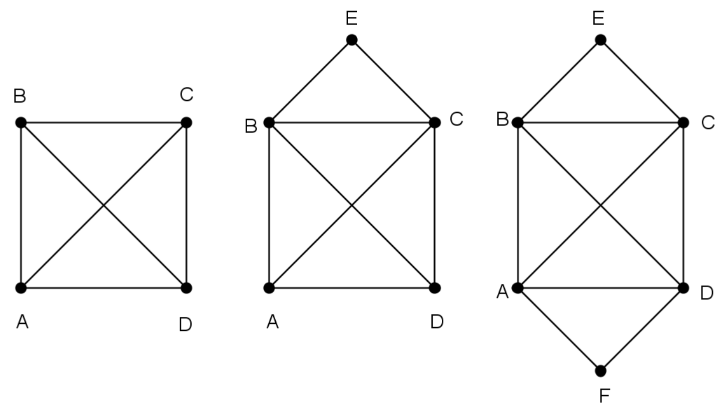
The Lesson: To start this lesson, it would be helpful to provide the background history of how Graph Theory started, beginning with the Königsberg bridge problem. By introducing the Königsberg bridge problem first, students will develop an intuition about methods that could be used to solve the Chinese Postman Problem. By presenting the Königsberg bridge problem as a graph, students will be able to understand how to consider trying to solve the problem. Dr. Bradley Carroll [8] provides students with a picture how the bridges look on a map, which can be reinterpreted as a graph as below.





The teacher should draw the above image of the graph on the board. Once the above picture has been drawn on the board, the teacher should have each student draw the graph on their own scratch paper if you have small white boards for each student to use this will encourage students to try and find different solutions. The teacher should ask students to try and solve the Königsberg bridge problem. Students should not be able to cross each bridge only once. Encourage your students to explore reasons they think it cannot be done and try other configurations to look for patterns.

To encourage students to discover the solution draw the following graphs provided by Sam Boardman, Tony Clough, and David Evans in their book *Decision Maths* [37]. The teacher should have the students determine if they can draw the graphs without lifting the pencil. The teacher should also have students make a note of which of the graphs can be drawn starting and finishing at the same vertex and which start and end at different vertices.



Have students discuss their results and come to an agreement on why the results are true.

Solutions: Graph 1 cannot be drawn without repeating an edge. Graph 2 can be drawn beginning at either base vertex and ending at the other. Graph 3 can be drawn starting and ending at any vertex of the graph.

Boardman points out that by looking at each of the graphs and looking at each of the vertices, students will be able to see if the graph is "traversable," which means that the graph can be drawn without having to take the pencil or pen off of the paper and without retracing any of the edges [37]. By looking at the order (number of incident edges) of the vertices we are able to understand the differences between the three graphs.

Vertex	Order
A	3
B	3
C	3
D	3

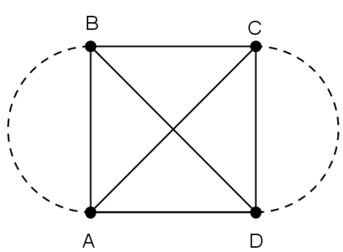
Vertex	Order
A	3
B	4
C	4
D	3
E	2

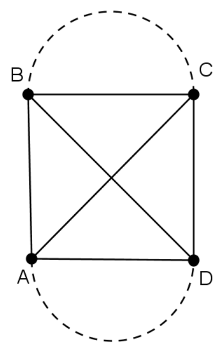
Vertex	Order
A	4
B	4
C	4
D	4
E	2
F	2

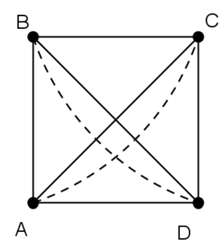
Boardman, Clough, and Evans describe the general solution: "[w]hen the order of all vertices is even, the graph is traversable and we can draw it. When there are two odd vertices we can draw the graph but the start and end vertices are different. When there are four odd vertices the graph can't be drawn without repeating an edge" [37].

A graph is said to be *Eulerian* if it can be drawn without lifting and without repeating edges, that is if all vertices must be even. If there are two vertices that are odd then the graph is considered to be *semi-Eulerian*, meaning that a trail can be drawn but the start and end vertices will be different [37].

If a graph has odd vertices, then some of the edges will need to be repeated.. A graph can be made Eulerian by adding necessary edges to increase the degree of the odd verticies as shown in the following examples [37].







Boardman, Clough, and Evans explain the connection between this process and solving the Chinese Postman Problem: "[T]o find a minimum Chinese postman route we must walk along each edge at least once and in addition we must also walk along the least pairings of odd vertices on one extra occasion" [37].

Boardman, Clough, and Evans provide the following algorithm for finding the optimal route.

1. List all odd vertices.

2. List all possible pairings of odd vertices.

3. For each pairing find the edges that connect the vertices with the minimum weight.

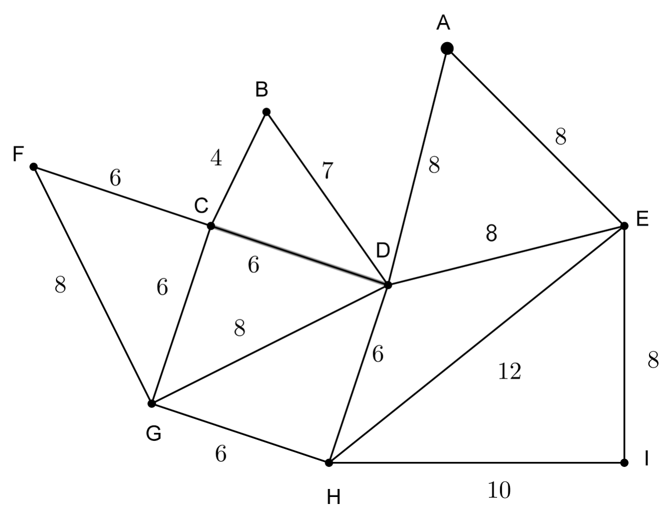
4. Choose a subset of the pairings such that the sum of the weights is minimized.

5. On the original graph add the edges that have been chosen in Step 4.

6. The length of an optimal Chinese postman route is the sum of all the edges in the modified graph.

7. A route corresponding to this minimum weight can then easily be found.

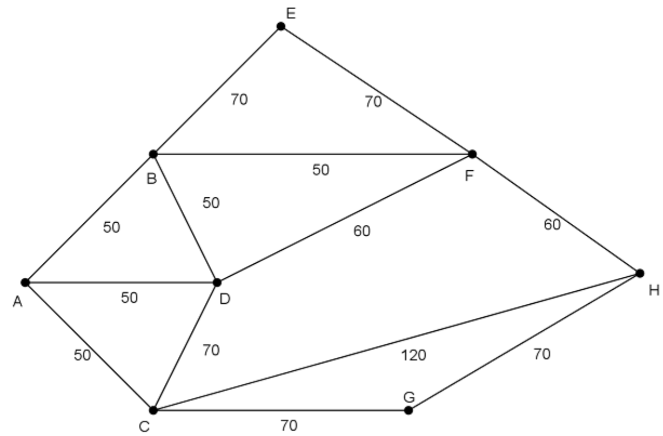
We now have a method for solving the original question posed, that is to find an optimal route for the postman in the following neighborhood.



1. Since there are no odd vertices we skip to step 6.

2. Making sure we do not cross any edge more than once. The postman travels along the following route: AEIHEDHGDCGFCBDA.

Boardman, Clough, and Evans provide the following example of a postman problem that contains odd ordered vertices: "In the following example, a postman has to start at A, walk along all 13 streets and return to A. The numbers on each edge represent the length, in metres[sic], of each street. The problem is to find a trail that uses all the edges of a graph with minimum length" [37].



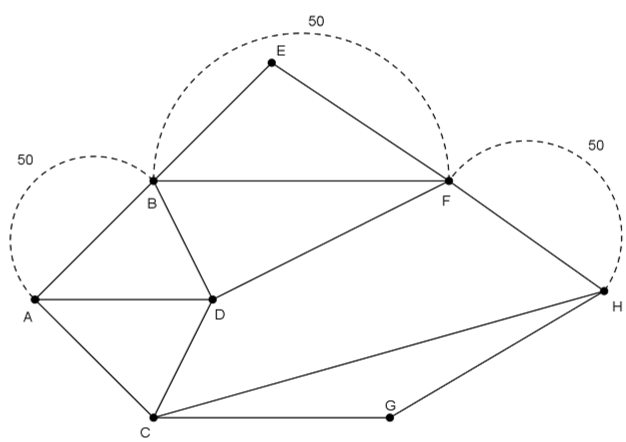
To solve the given problem the following steps using the Chinese postman algorithm.

1. The odd vetices are A and H.

2. There is only one way of pairing these odd vertices, namely AH.

3. The shortest way of joining A to H is using the path AB, BF, FH, a total 160.

4. Draw these edges onto the original network.



5. The length of the optimal Chinese postman route is the sum of all the edges in the original network, which is 840m, plus the answer found in Step 4, which is 160m. Hence the length of the optimal Chinese postman route is 1000m.

6. One possible route corresponding to this length is ADCGHCABDFBEFHFBA, but many other possible routes of the same minimum length can be found.

The evaluation for this lesson will be in the form of worksheets such as those provided in Appendix C. Now that the students have spent time going over how to solve graphs with even ordered vertices as well as both even and odd ordered vertices, hand out worksheets. The teacher should allow students to work alone or in pairs for the remainder of the class time. Provide students with plenty of time to work through the problems, before the day ends bring the students back together and go over the answers to the worksheet.

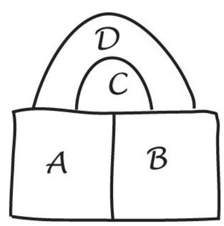
Possible extensions for the students: (1) Draw a graph which has exactly seven vertices of odd order and then try to find an optimal Chinese Postman Problem path. Note: this is not possible. Every graph must have an even number of odd vertices. By asking your students to attempt this task, the students should be able to determine that it is impossible and also explain why. This is one of the first theorems in any Graph Theory text, but is one which students at this level should be able to "discover" and "prove" on their own. (2) Make their own Chinese Postman Problems to exchange with other students. Instruct your students to write down the steps to solve for a solution for their own graph. Once the students have finished have them switch with other students within the class and allow the students to solve each others graphs and verify that the provided solution is correct. (3) Another possible extension of the Chinese postman is a special case, the Rural Postman Problem, which focuses on arc routing and optimizing the minimum cost to travel all of the arcs.

3.2 The Four Color Problem

Our second Graph Theory problem is the Four-Color Problem, which states that every planar graph has a chromatic number less than or equal to four. David Burton explains in nontechnical terms in his book, "The History of Mathematics: An Introduction", that the Four-Color Theorem can be stated as: “any conceivable map drawn on a plane or on the surface of a sphere can be colored, using only four colors, in such a way that adjacent countries have different colors” [6]. Ed Wheeler and Jim Brawner give an explanation of why the Four Color Theorem must be true in their textbook, “Discrete Mathematics For Teachers”: “since every map in the plane can be colored using the coloring of its dual planar graph, this verifies the conjecture that every map can be colored using at most four colors” [48].

3.2.1 History of the Four Color Problem

The first mention of the Four-Color Theorem occurred in 1852 and was considered only as a problem. It was first expressed in a letter sent by Augustus De Morgan, a professor of mathematics from University College of London, to Sir William Rowan Hamilton, a celebrated Irish mathematician and physicist. In the letter, preserved in the Trinity College Dublin archives, De Morgan wrote,

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact–and do not know yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured–four colours may be wanted, but not more–the following is his case in which four are wanted.
>
> 
>
>
> Query cannot a necessity for five or more be invented . . . What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England . . . The more I think of it the more evident it seems.

In the 1880s when the Four-Color problem had become more popular, Frederick Guthrie, as student of De Morgan, explained that he had gotten the question from his brother Francis Guthrie who had attended De Morgan's class in the 1850s. Francis Guthrie noticed that he only needed four colors to color a map of England. Francis asked his brother, Frederick, to ask De Morgan if this had been proven mathematically [6]. Robin J. Wilson, Norman Biggs, and E. Keith Lloyd write in their book, "Graph Theory: 1736-1936", that Francis Guthrie, who by the 1880s was a mathematician in South Africa, had shown his brother Frederick "the greatest necessary number of colours to be used in colouring a map so as to avoid identity of colour in lineally contiguous districts is four" [32]. Francis had given Frederick permission to send the theorem to De Morgan. De Morgan took the problem as a new problem and gave Guthrie credit, even though De Morgan did not feel that the proof that Francis Guthrie had given was "altogether satisfactory to himself" [32]. Francis Guthrie is credited with coming up with the Four-Color problem, and De Morgan is responsible for the subsequent spread of the Four-Color problem. Wilson, Biggs, and Lloyd further explain "[h]is [De Morgan] curiosity had been aroused by Guthrie's question, and he spoke of it to other mathematicians and students, so that the problem became part of mathematical folk-lore" [32]. William Hamilton, the recipient of De Morgan's letter, did not work on the problem which he considered of little importance [6].

For nearly a century, the Four-Color Conjecture was one of the most popular problems in all of mathematics [6]. Many mathematicians tried to prove the conjecture to be true, but the proofs were later found to have errors. In 1878, Arthur Cayley presented the problem at the London Mathematical Society, which sparked the interest of Arthur Kemp, a lawyer, who was also a member of the London Mathematical Society [6]. Kemp published his proof in 1879 in the newly established mathematics journal, *American Journal of Mathematics*. In the paper, Kemp believed that he had proven for any map on a sphere only four colors are needed to color the map. For more than a decade, the proof was held as true [6]. However, in 1890, Percy Heawood found a mistake in Kemp's proof. Heawood published a "simplification of Kemp's proof that each map drawn on the plane or sphere can be colored by at most five colors, the Five-Color Theorem" [6]. The Four-Color Conjecture continued to gain the attention of many mathematicians over the years however the proof of the Four-Color Conjecture eluded many mathematicians who sought to solve the problem. It was not until 1922 that a giant leap towards the proof of the Four-Color Conjecture took place Burton writes, "in 1922 when it was shown that an arbitrarily drawn map of 25 or fewer countries is four-colorable; thus, any counterexample to the conjecture would have to be a map of at least 26 countries” [6]. The number of countries continued to grow until it reached 96 countries, at which point the “results became superfluous” [6]. In 1976, Kenneth Appel and Wolfgang Haken of the University of Illinois provided a proof of the Four-Color Conjecture using a computer, at which point the conjecture became widely accepted as the Four-Color Theorem. According to Edward Swart from the University of Waterloo, “the final working list of 1482 unavoidable configurations embraces all the unavoidable configurations arising from their discharging procedure. Thus, the Haken and Appel proof is subject to some degree of uncertainty” [44]. Appel, Haken and Koch are the first people to introduce the use of computers into solving mathematics.

3.2.2 The Four Color Problem in the Elementary Classroom

Students as early as Kindergarten are able to learn the basics of the Four Color Problem. The Four-Color theorem even allows students to develop mathematical proofs without even knowing that they are doing so. The teacher should be able to encourage and celebrate students’ success in solving something as easy as coloring a map with only two colors. Nancy Casey and Michael R. Fellows write in the book, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science: Discrete Mathematics in the Schools*, that “[a] teacher not equipped with the idea of the importance of mathematical proof and expecting to encounter and develop this concept, is not equipped to fully appreciate and empower the problem-solving going on” [10].

The following lesson plan can be done in one or two days. The teacher should start off the lesson by teaching the students the history behind the Four-Color problem. Once students have been given time to be introduced to the problem, the teacher should tell them that the goal for the day will be to try and color a map of the United States of America with as few colors as possible.

The main objectives for the lesson will be:

1. Coloring a map with as few colors possible
2. Using logical reasoning to explore patterns
3. Learning and applying problem solving skills

The NCTM Standards which the assignment will cover are:

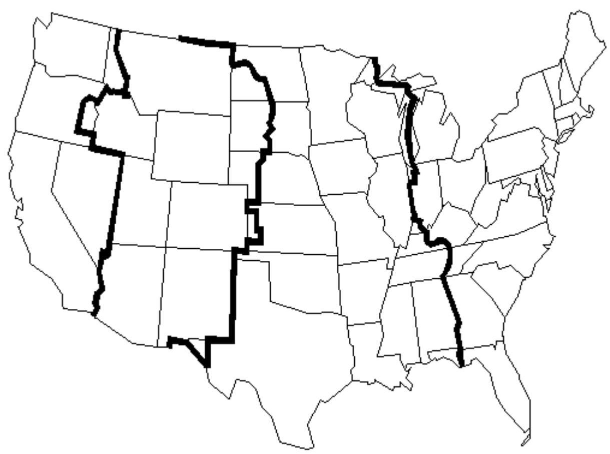
Grades Pre-K-2 Standards 1, 2, 3, 5, 6, 7, 8, 9, 10

Grades 3-5 Standards 1, 2, 3, 5, 6, 7, 8, 9, 10

Consider for example Standard 6.a. "Build new mathematical knowledge through problem solving," for both Pre-K-2 and Grades 3-5 Standards. The Four-Color problem allows students to be able to learn how to problem solve when given an object to color with as few colors as possible. Students are able to develop the problem solving and critical thinking skills needed to be successful in future mathematics courses while working on problems such as the Four-Color Theorem.

Consider another standard that the Four-Color problem addresses, Standard 9.c. "Recognize and apply mathematics in contexts outside of mathematics." The Four-Color problem allows students to make connections of how mathematics can be used in scheduling classes, meetings, flights, and shipment of certain chemicals. Students will be able to understand the how abstract mathematics can be applied to real world problems.

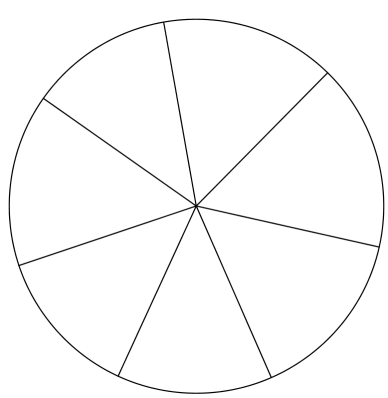
The Lesson: Once the Students have been introduced to the history of the Four-Color Theorem have the students consider the following problems to help reinforce Gutherie's Conjecture.



Richard Francis in his book *The Mathematician's Coloring Book* suggest having the students start off coloring the time zones of the United States of America. Francis provides the following directions and questions to ask students [18].

1. Color the time zones above so that no two zones of the same color touch along a border.
2. Did you need as many as four different colors?
3. Could the shading have been done with only two colors?
4. Explain why time zones exist. How many time zones does the earth have?

The teacher should have the students consider the following picture. The teacher should provide the students with several slices of pie.

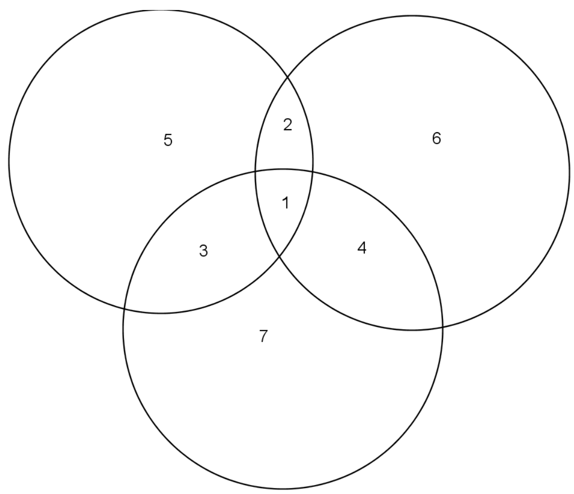


1. Color the slices of pie with seven different colors (at least four of each color).

2. Use the fewest colors to create a whole circle, so that no two pieces of the same color share a side. Note that pieces of the same color may touch at a single point, but they are separate pieces.

3. Could the pie have been created with only three colors?

Once the students have started to master the concepts of trying to use as few colors as possible have the students try the following example provided by Francis.



Have students "[c]olor the seven regions above so that no two of the same color touch along the border. Use the LEAST NUMBER OF COLORS possible. Remember that two like colors may touch at a single point" [18]. Once the students have mastered the previous examples, hand the students the following map of the United states and have the students color the map with the least number of colors possible.

Allow the students to have plenty of time to color the map of the United States. Some students may need to start over with a blank map if they make a mistake. Allow for the remainder of the time for students to work on the assignment bringing the class back together at the end of the class to discuss the results that had discovered. Students may struggle with the states Utah, Colorado, Arizona, and New Mexico. It maybe helpful to have the students start with these states and work from there.

This can also be done with a county map of the state in which the students live. Students can also be asked where the Four-Coloring Theorem could be applied in the real world. Shamin Ahmed provides several suggestions for students to consider on how the Four-Color Theorem us used in real life situations. Ahmed's paper, entitled "Applications of Graph Coloring in Modern Computer Science", lists guarding of an art gallery, postal sorting, round-robin sports scheduling, air craft scheduling, biprocessor tasks, radio wave frequency assignments, final exam scheduling, map coloring, and Groups Special Mobile (GSM) phone networks as real world problems [1].

Once students understand the basic idea of coloring a map other extensions can be explored that incorporate other aspects of mathematics. Teachers can create or find material online for coloring games that have both weighted and non-weighted areas. Elementary students will use logic and reasoning to color a graph or picture with a select number of colors or to find other patterns in a given graph.

Students can be introduced to probability and deductive reasoning by playing coloring games that also implement the use of dice examples, which can be seen in Appendix C. When the teacher allows students to play coloring games, the students are able to develop game strategies for forcing the other player to use a color or creating a situation where the other player cannot use a color. The Chinese Postman Problem and the Four-Color Theorem provide students with the necessary background to become problem solvers and critical thinkers. Each problem introduces students to abstract mathematics and the theory behind their solutions, as well as to how to find applications of advanced mathematics in the real world.

Chapter 4

Applications of Combinatorial Game Theory in the Classroom

Mathematical games can be a useful tool for teachers of the K-12 classroom. By incorporating mathematical games into the curriculum, students are able to take a break from the normal class time and spend an hour or so playing mathematical games while unknowingly developing the logic, deductive and inductive reasoning, and problem solving skills used in mathematics. Gillian Hatch lists in her article, “Using Games in the Classroom”, the added benefits of playing games. Hatch says “[a] game can generate an unreasonable amount of practice. Geometric games create a context for using geometric reasoning. A game will often result in the making of generalised [sic] statements. A game can allow the introduction of ideas that are difficult to develop in other ways. Games seem to be able to lead pupils to work above their normal level” [25]. Mathematical games such as Tic-Tac-Toe (TTT) and the Mathematical game, SET® can be played by students of all ages. Both of these games can be used to introduce abstract and advanced mathematics to students at a young age. They also have a deep connection to two mathematical structures that will be discussed first, those of Latin Squares and Affine Planes.

4.1 Latin Squares

Maureen T. Carroll and Steven T. Dougherty explain in their article,"Tic-Tac-Toe on a Finite Plane", that "it is natural to explain the game of tic-tac-toe on a finite plane by the connection between planes and these [Latin] squares" [they] begin with "an explanation of Latin squares, affine planes, and the relationship between them" [9]. Yul Inn gives a description of a Latin square, from his paper titled "Latin Squares For Elementary and Middle Grades," as a "combinatorial object whose definition is based on very simple concepts and logical conditions" [28]. Inn explains that "[t]he size n is called the *order* of the Latin square. The representation of a Latin square is normally a square grid of *cells*, each cell containing a symbol" [28]. Inn noted that the symbols in a Latin square can be anything that the teacher wants to use. The majority of Latin square examples use the "first n natural numbers or the first n letters for the symbols in Latin squares of order n" [28]. If the teacher chooses the first n whole numbers for the symbols then we can formally define a Latin square the same way Carroll and Dougherty do, which is "[a] Latin square of order n is an $n \times n$ matrix with entries from $n = 0, 1, 2, \ldots, n-1$, where each number occurs exactly once in each row and each column" [9].

In his paper, Inn provides an example of a Latin square of order 4 which is a 4×4 matrix using the symbols α, β, χ, and δ. Notice that every symbol appears exactly once in each row and exactly once in each column.

α	β	χ	δ
χ	δ	α	β
δ	χ	β	α
β	α	δ	χ

The following examples are examples of Latin squares of orders 2, 3, and 4, each using the first n whole numbers.

0	1
1	0

0	1	2
2	0	1
1	2	0

0	1	2
1	2	0
2	0	1

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

Students are familiar with seeing partial Latin squares in games like Sudoku. Inn explains that “[a] *partial* Latin square is an $n \times n$ grid of cells in which some cells are filled with symbols, and no symbol occurs more than once in any row or column [28]. Here is an example of a partial Latin square and a solution provided by Inn [28].

A

1	**2**	
	1	

B

1	**2**	3
2	3	1
3	**1**	2

Inn shows that the “[p]artial Latin square A can be completed to form Latin square B” [28]. However, as noted by Inn some partial Latin squares cannot be solved. Inn provides this example [28].

1	**2**	
		3

Since a 3 cannot be placed in the upper right-hand corner, the Latin square cannot be completed. Inn explains, having students work partial Latin squares helps students develop logical reasoning [28].

Inn provides three basic strategies for solving Latin square problems.

> *Row-forced or Column-forced entries:* If all but one cell of a row or column is filled, the symbol in the remaining cell must be the one remaining symbol.
>
> *Row-and-column-forced entries:* For a given cell, if all symbols except one appear in either the row of the cell or the column of the cell, then the one remaining symbol must be in the cell.
>
> *Last-cell-forced entries:* If a symbol occurs in all but one of the rows (and columns) its last occurrence is forced to be in the row and column in which it does not yet occur.

Inn notes that "the first type of reasoning is actually a special case of the second, but it is so much easier to identify the first case that it makes sense to call it out separately" [28]. Students can practice solving partial Latin squares with the worksheet provided in Appendix D.

Once students are comfortable with completing a Latin square, they should next be introduced to the idea of Mutually Orthogonal Latin squares. The following definition is provided by Carol T. Benson, Kyle P. King, and Jeffry A. Mudrock, in their paper "An Introduction to Discrete Mathematics in the Classroom: Latin Squares." Mutually orthogonal Latin squares (MOLS) are "[t]wo Latin squares of order n, which, when superimposed, form each of the n^2 possible ordered pairs of n symbols exactly once. At most, $n - 1$ Latin squares can be mutually orthogonal." That is,"[t]wo Latin Squares of order n with labels 1 through n are mutually orthogonal if, when superimposed over one another, they yield exactly one of each ordered pair {(1, 1), (1, 2), . . . , (1, n), (2, 1), (2, 2), . . . , (2, n), . . . (n, 1), (n, 2), . . . , (n, n)}" [7]. Benson, King, and Mudrock provide the following example of a pair of orthogonal Latin squares.

1	2	3
2	3	1
3	1	2

is orthogonal to →

1	3	2
2	1	3
3	2	1

To understand why these two Latin squares are orthogonal, the teacher should encourage the students to combine the two Latin squares by superimposing the two squares on each other, once the students have completed superimposing the two Latin squares, have the student verify that they got the following square.

(1, 1)	(2, 3)	(3, 2)
(2, 2)	(3, 1)	(1, 3)
(3, 3)	(1, 2)	(2, 1)

The teacher should point out to the students that since no ordered pair occurs twice, the two Latin squares are mutually orthogonal. Once students understand what it means for two Latin squares to be orthogonal the teacher can extend that understanding to larger groups of Latin squares. In general the teacher can say a set contains Mutually Orthogonal Latin Squares (MOLS) if every pair of Latin squares in the set are orthogonal to one another. MOLS will play an important roll in the extension of Tic-Tac-Toe we will cover in the next section.

Latin squares can also be used when working with geometry. Padraic Bartlett explains in a lecture, given in 2012 at a math camp at the University of California Santa Barbra, how to use Latin squares to do geometry. Bartlett starts off by providing a definition of an affine plane:

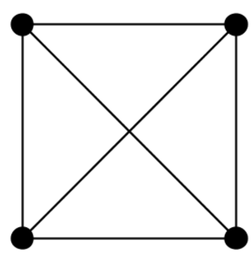
"**Definition.** An **affine plane** is a collection of points and lines in space that follow the following fairly sensical rules:

(A1) : Given any two points, there is a unique line joining any two points.

(A2) : Given a point P and a line L not containing P, there is a unique line that contains P and does not intersect L.

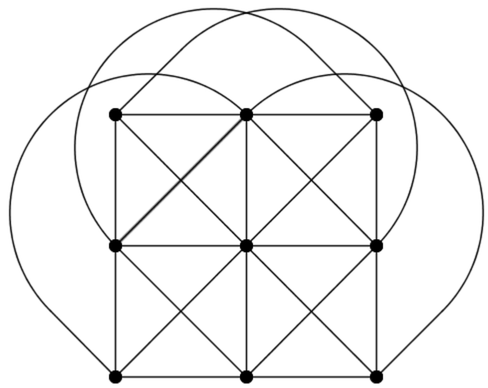
(A3) : There are four points, no three of which are collinear. (This rule is just to eliminate the silly case where all of your points are on the same line.)" [2]

Bartlett also provides an example of an affine plane, our familiar coordinate plane $\mathbb{R}^2$, but further explains that when connecting affine planes and Latin squares, the only affine planes that will be considered are finite affine planes [2]. According to Bartlett, a finite affine plane is a plane with a finite number of points [2]. Below is an example provided by Bartlett for a finite affine plane of order 2. Bartlett states that "the following set of four points and six lines defines an affine plane" [2].



The above affine plane fulfills the definition. Since: (A1) choosing any two points, those two points form one of the six lines. (A2) if for example, the student chooses the top left point and then chooses the vertical right side of the box line then (A2) is satisfied by the vertical left side. It should be noted that the two diagonal lines are not intersecting as two lines can only intersect at a point. They appear to cross in this depiction, but they do not intersect one another. Then for (A3) no matter what three points the students choose, none of them will be collinear. Therefore the above picture is an affine plane.

Another example of an affine plane that contains nine points and twelve lines is shown in the picture below:

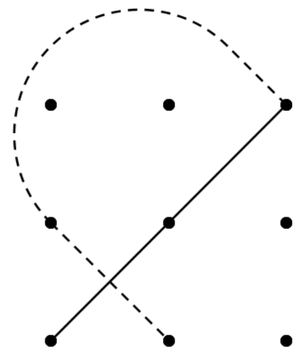


Bartlett explains that "[a]ll lines in the picture above contain three points" [2]. There are four curved lines, two diagonal lines, and there are six lines that go up and down or left and right.

Bartlett also mentions several properties that every finite affine plane satisfies [2]:

- There is an integer n such that every line in the plane contains exactly n points, and every point lies on precisely $n + 1$ lines. We call this value the **order** of the plane.
- There are exactly n^2 points in the plane.
- There are exactly $n^2 + n$ lines in the plane.
- The lines can be partitioned into $n + 1$ distinct parallel classes each containing exactly n non-intersecting lines.
- Two lines from distinct parallel classes intersecting at a unique point.

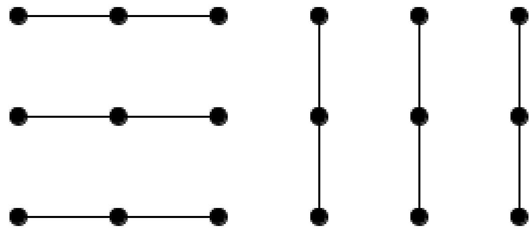
To understand each of these properties, the teacher should have students explore the previous example and have the students identify the points, lines and parallel classes of the plane. Next the teacher should have the students choose two lines from different parallel classes and find the point of intersection. One possible example would be the following lines:



The teacher should point out that the lines from the separate parallel classes intersect at the top right hand corner at that unique point. It is important that the teacher stress to students that while these lines appear to cross one another in the lower left, two lines can only intersect at one of the points of the plane. This will take students some time to understand so the teacher should encourage students to consider example pairs from several parallel classes and have them circle the unique **point** of intersection.

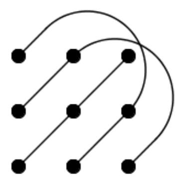
To see the connection between Latin squares and affine planes, the teacher should have the students consider the collection of cells containing a single symbol in the Latin square as representing the points in a line of the plane. The collection of such sets (lines) in a single Latin square represents a parallel class of the plane. The other parallel classes are likewise formed using MOLS. Bartlett provides the following examples for the process where n is 3.

Consider for example $AG(2,3)$ (the affine plane of order 3), and its parallel classes: vertical and horizontal lines are always parallel classes of an affine space.

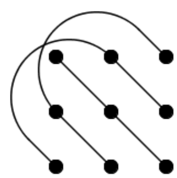


From a pair of MOLS, the following parallel classes are formed

1	2	3
2	3	1
3	1	2

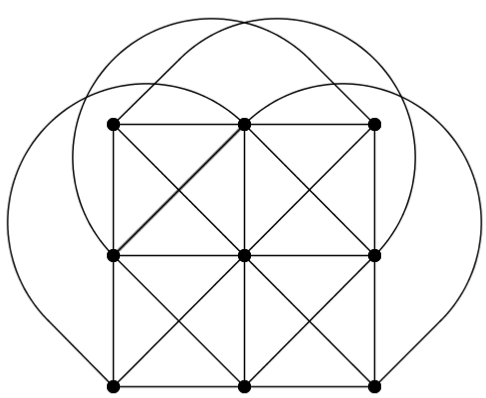


1	2	3
3	1	2
2	3	1



When considering all the lines of the four parallel classes together, the affine plane is created as shown below.

Affine plane:



Carroll and Dougherty explain how to play Tic-Tac-Toe (TTT) on an affine plane where "winning lines are prescribed by the geometry of a finite affine plane" [9]. Carroll and Dougherty continue to explain that "these new lines make it more difficult to identify a win here than in the standard game ... the reasons for this complexity is that lines in an affine plane need not to appear straight" [9]. The abstract nature of this new game helps students to develop high-cognitive and rational thinking. Carroll and Dougherty claim that "[t]he geometric intuition required to understand finite planes often proves elusive, as our Euclidean-trained minds have preconceived notions of lines and points. The new version of Tic-Tac-Toe helps to develop this intuition. Moreover, this game relates geometric concepts to game-theoretic concepts as the natural question of winning strategies arises" [9]. Carroll and Dougherty explain that reviewing how Latin squares and affine planes work is needed for students to be able to grasp and understand what constitutes as a winning line [9]. Since TTT can be played on a finite plane, as Carrol and Dougherty explain, "it is only natural to explain the game ... by the connection between planes and these squares" [9]. By looking now at the TTT board as a set of Latin squares, students are able to see the connections between points. Carroll and Dougherty recall that "[o]n a finite plane, each line must contain the same number of points and each point is incident with the same number of lines. The number of points on each line is called the order of the plane" [9]. In a typical TTT board, there are nine points and eight possible lines, when playing on the affine plane the player has nine points, but now has twelve possible lines. Increasing the number of lines increases the complexity of game play and forces students to reconsider a game that they have long understood.

We will see a connection to Latin squares and affine planes in both of our example games. These lessons can be taught either on their own, or by beginnning with a short unit on Latin squares and affine planes. The two games we will consider are Tic-Tac-Toe and SET®.

4.2 Tic-Tac-Toe

Tic-Tac-Toe (TTT) is a two player game where each player takes turns playing their pieces (usually X's and O's) on the game board in an attempt to get three in a row. Dan Garcia who helped to create Game Crafters, a research group from UC Berkley, describes the board as follows: the "game board is made by drawing two parallel vertical lines, then dividing the 2 lines into thirds with two horizontal lines. The end result should be a grid with nine equal squares" [19]. As Elwyn Berlekamp, John H. Conway and Richard Guy remark in their book *Winning Ways for your mathematical plays* "[w]hen neither player is able to make a line we have a tied game" [15]. Berlekamp, Conway and Guy went on to say that most children learn quickly how to cause a tie in a game [15].

4.2.1 History of the Tic-Tac-Toe

The game Tic-Tac-Toe (TTT) dates back to ancient Egypt. TTT has been found drawn on temples in Egypt and medieval churches in England [19]. Even though TTT is generally played by children, children did not always play the game. Garcia explains the history of TTT as being a "game that was linked to pagan rituals dedicated to the magic properties of the nine-square grid. The grid was known as the Magic Square because the numbers 1-9 could be arranged so that their sum is the same horizontally, diagonally or vertically" [19]. During the Middle Ages the Magic Square was believed to give an understanding of the world using numbers. Gracia also points out that during this time "the Magic Square was known by secret societies as the Cabala of the Nine Chambers" [19]. These secret societies believed that Magic Squares held hidden messages about the world; however, today Garcia explains that "the game is more known as a simple childhood pastime" [19]. Garcia describes how TTT received its name, by providing a history of TTT and explaining that in the 16th century TTT was originally called "Tit-Tat-Toe" where, "[t]it by itself means to slap and a "tit for tat" is retaliation. Toe, which is the third piece placed makes the winning combination by securing the other two pieces" [19].

4.2.2 Tic-Tac-Toe in the High School Classroom

Students who have played TTT for any amount of time realize that there are successful strategies that TTT players can use to begin to create and cause tie games. As a first step in extending the students understanding of TTT, the teacher should have students consider the common board re-imagined as a collection of points and lines as in Euclidean Geometry, with nine points instead of boxes. Once the students have gotten used to the new game board the teacher should suggest adding a few changes to add another degree of difficulty to the standard TTT game.

This lesson should be able to be covered in one or more class periods, depending upon the students level of mathematics maturity and previous background knowledge. The main objectives for this lesson will be:

1. Students learn to play TTT on various boards.
2. Students develop higher level and rational thinking skills.
3. Students begin to understand new topological surfaces.

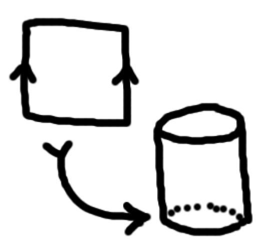
The NCTM Standards this lesson will address, which can be found in Appendix A, will be:

Grades 9-12 Standards 2a, 2d, 3, 6, 7, 8, 9, 10

Consider for example standard 3.a that this lesson will address: "Analyze characteristics and properties of two-and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships." The student will be looking at both a two-dimensional game board as well as a three-dimensional game board. The students will have to be able to reason and develop mathematical arguments for why they have a winning line and why the other player does not. By learning about playing on boards such as $AG(2,3)$ students will be able to learn about geometric relationships.

Another standard to consider that this lesson addresses is the standard 8.c: “Analyze and evaluate the mathematical thinking and strategies of others.” Here students will again be playing games and developing their own winning strategies in order to beat the other player. By learning how to create a winning line in $AG(2,3)$, students will be able to develop successful winning and blocking strategies when playing TTT in an affine plane.

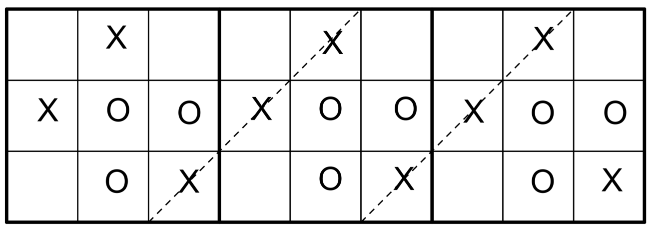
The Lesson: The teacher should start off the assignment by giving the students the history of how TTT was developed. Once the students have been introduced to the history of TTT, the teacher should have the students break up into pairs and allow them to play traditional TTT for a few minutes. Once students have spent time playing TTT, the teacher should next have the students discuss why playing traditional TTT becomes boring after a few games. The teacher should ask the students what would happen if they were to change the surface on which the traditional TTT game was played and if the outcome would still be the same? The Math Explorers' Club at Cornell University is a group supported by the National Science Foundation (NSF). This group of students created a website with resources for junior high and high school teachers to use, exposing students to more advanced topics in mathematics [12]. The first surface that The Math Explorers' Club suggests to play TTT on is a cylindrical board. The students still play TTT on a normal TTT board; however the board now has two sides that are considered attached, the left and the right side of the board [12]. The Math Explorers' Club explains that by considering the left and the right sides attached, the game now has increased the number of ways to win by four [12]. The game board went from having eight possible winning lines to now having twelve winning lines.



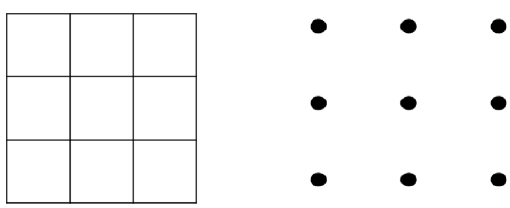
For example, suppose the game below has been played on a traditional TTT board. Both players would see that this game will be a cat's game or a draw where neither player has a winning line. However, if the game is being played on a cylinder, then the player who represents "*X*" actually has a winning solution.

	X	
X	O	O
	O	X

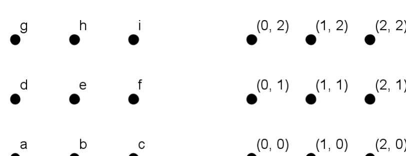
The TTT board below shows one way to visualize the above game played on a cylinder. The winning line should be obvious when drawing the boards side by side, where the original game is in the middle.



The teacher should remind students about affine planes and the parallel classes that are associated with affine planes of order 3. The teacher may also find it useful for the students to treat each of the boxes in a traditional TTT board as points rather than boxes as shown below.

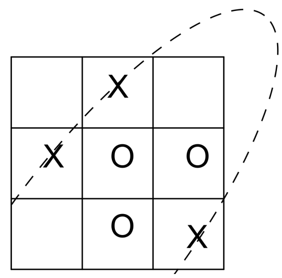


The teacher may also either label the lattice of points with letters or with the coordinates that are typically associated with graphing in a Cartesian coordinate system.

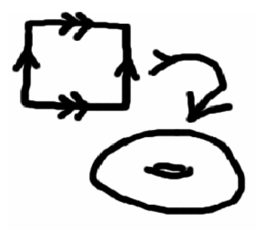


Once the teacher has reviewed Latin squares and affine planes, the teacher should have the students write all the possible winning lines for the affine plane of order 3. Carroll and Dougherty provide the following solutions for all of the winning lines in $AG(2,3)$, "where P={a, b, c, d, e, f, g, h, i} and L={{a, b, c}, {d, e, f}, {g, h, i}, {a, d, g}, {b, e, h}, {c, f, i}, {a, e, i}, {c, e, g}, {a, g, f}, {g, b, f}, {i, b, d}, {c, h, d}}" [9].

If the teacher would like to use this lesson as an opportunity to teach about or to reinforce information on modular arithmetic the teacher should use the Cartesian points, and take some time discussing modular arithmetic. Again the initial goal for the students should be to identify all of the winning lines for the configuration of points. The teacher may want to have the students consider the previous cylinder board problem where the $X's$ are located at the points $(0,1)$, $(1,2)$ and $(2,0)$, when discussing using modular arithmetic while using coordinates to play TTT. Since many students may not completely grasp the concept of winning lines in $AG(2,3)$, the teacher should ask the students at what positive coordinate should the next X be placed after $(0,1)$ and $(1,2)$ so that all three points would lie in a straight line if this game was played on an infinite lattice of points? Students should provide the following answer: $(2,3)$. After indicating that this is correct, the teacher should ask the question "What is that coordinate mod 3?" Students should be able to remember that since the last coordinate is a 3 in mod 3 this value is actually a 0, making the coordinate $(2,0)$ instead of the original coordinate of $(2,3)$. Notice that students may also respond with other coordinates that lie on the same line. For example, a student may give the point $(3,4)$. By looking at what those coordinates are mod 3, we see that $(3,4)$ reduces to $(0,1)$ and $(0,1)$ is already one of the original two points on the line. A few other points that the students can give are $(4,5)$, which reduces to $(1,2)$ mod 3, and the point $(5,6)$ which reduces back to $(2,0)$. The teacher should inform the students that this process will continue since the points are being reduced mod 3. Once the students are able to identify the lines in the coordinate plane, the teacher should go back to the original cylinder problem and have the students draw the winning line for the affine plane of order 3.



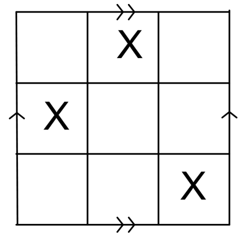
Now that the students have mastered how to play TTT on a cylinder, have the students consider playing on a torus. The Math Explorers' Club explains in order to play on a torus the sides must be joined and the top and the bottom edge must be joined as well so that board will look like the following [12].



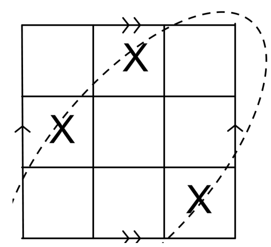
The teacher should begin to pose the following questions as suggested by the Math Explorers' Club [12]:

- Are there new ways to win toroidal TTT compared to standard TTT? How about compared to cylindrical TTT?
- How would a game of chess work out on a torus if we started from the usual position?
- Can you come up with better starting positions for toroidal chess?

The teacher should place the following TTT board on the board and ask the students to consider what the board looks like on a torus:



The students should be able to start to see that, even though on a traditional TTT board the player who is “$X's$” would not have a winning line, since the game is being played now on an affine plane. In $AG(2,3)$, however; a winning line does exist as shown below:



Now that students have an idea of how to play on a torus, allow them to play a few games with a partner. The teacher may want to have a picture of all the winning lines of $AG(2,3)$ on the board where the students can see all the possible lines or provide a handout to each pair of students with all the possible winning solutions on it. As students are playing, the teacher may ask the question, “Is there any advantage in who goes first?” Once the students have played a few games, bring the students back together and ask them if they preferred playing on new boards instead of playing on a standard TTT board and what they learned. The teacher might ask students if they prefer playing on the “cylindrical” TTT board or if they have actually begun playing on the point/line model of the affine plane instead.

Once the students have become comfortable playing on $AG(2,3)$, the teacher can have the students expand the board so that the students are now playing on the board $AG(2,4)$. Students should recall that the order of the board is the number of points on any given line as well as other information about the plane. The teacher should have the students recall that the new board will be a 4×4 TTT board, with 16 points (n^2) and 20 lines ($n^2 + n$). Where 4 lines will be horizontal, 4 lines will be vertical, and the remaining 12 lines will be identified by referring back to the $n - 1$ MOLS of order n [9]. Carroll and Dougherty point out that the teacher should have the students "[r]emember , each of the MOLS of order n defines n lines (displayed as identical symbols). Since there are $n - 1$ MOLS, each defining n lines, we have our remaining $n(n - 1)$ lines" [9]. Carroll and Dougherty provide the following two TTT games:

X_1	X_4	
X_3	O_2	O_1
O_3		X_2

X_1	X_2	O_3	X_3
O_1	X_6	X_4	
X_7	O_6	O_2	O_5
O_4	X_5		O_7

In the first game "X" has a winning line "on the affine plane of order 3 since $\{X_2, X_3, X_4\}$, [The underscored number represents $X's$ second, third, and fourth move], forms a line" [9], where as in the second game "O" has a winning line on the affine plane of order 4. Both of these wins can also be verified by referring back to the Latin squares as shown below.

0	1	2
2	0	1
1	2	0

0	1	2
1	2	0
2	0	1

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

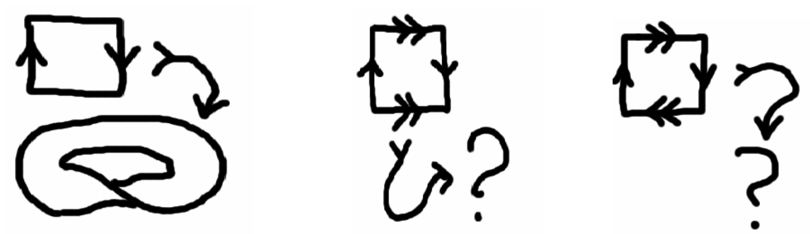
0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

The teacher should point out that the second Latin square of order 3 verifies the winning line. Where as the second game on the plane of order 4 has the following winning line $\{O_1, O_3, O_6, O_7\}$ and can be verified by the second Latin square of order 4 in the picture above using the cells marked "2" [9].

After the teacher has introduced playing on the two planes $AG(2, 3)$ and $AG(2, 4)$, the teacher should allow students to play on both planes. The teacher may find it helpful to provide a copy of the handout of the MOLS, which can be found in Appendix D. By providing the students with the Latin squares handout, they will be able to verify a win on either $AG(2,3)$ or $AG(2,4)$.

Possible extensions of playing TTT on different boards would be to have the students play on a Mobius Strip, Klein Bottle, larger affine planes or even a Projective plane. The Math Explorers' Club provides the following images for the students to consider for each of the above mentioned geometric objects [12].



As students begin to play TTT on various boards, students will begin to think and reason more abstractly and mathematically since the new winning lines are no longer straight lines as they were on the standard TTT board. The students cognitive learning will broadened by being introduced to higher level mathematics at an early age. Students will be able to use Latin squares to find winning solutions or the students will have the ability to use modular arithmetic to find the winning solutions. Another interesting game that has interesting connections to $AG(2,3)$ is the game SET®.

4.3 The mathematics of SET® with various restrictions

The game of SET® is played with a special deck of 81 cards. The official rules for the game of SET® are as follows:

The objective of the game is to identify a **set** of 3 cards from 12 cards laid out on the table. Each card has four features, which can vary as follows:

NUMBER: Each card has *one, two or three* symbols on it;

COLORS: The symbols are *red, green, or purple*;

SHADING: The symbols are either *filled in, outlined or stripped*.

SHAPE: Each card has *ovals, squiggles or diamonds* on it;

A **set** consists of 3 cards in which each of the card's features, looked at one-by-one, are the *same* on each card, or, are *different* on each card. All of the features must separately satisfy this rule. In other words: *shape* must be either the same on the 3 cards, or different on each of the 3; *color* must be either the same on the 3 cars, or different on each of the 3; etc.

How the game is played: The dealer shuffles the cards and lays 12 (in a rectangle) face up on the table so that they can be seen by all. Players remove **sets** of 3 cards from anywhere on the table. Each **set** is checked by the others. If correct, the **set** is kept by the player and the dealer replaces the 3 cards with 3 from the deck. Players do not take turns but pick up **sets** as soon as they see them. A player must call **set** before picking up the cards. After he/she has called **set**, no other player can pick up the cards until that player is finished. The **set** must be picked up within a few seconds after calling it. If a player calls **set** and does not have one, he/she loses one point, and the 3 cards are returned to the table.

If all players agree that there is no **set** in the 12 cards, 3 more cards are laid face up (there are $\sim$33:1 odds that a **set** is present in 12 cards, and $\sim$2500:1 odds when 15 cards are on the table). The 3 cards are not replaced when the next **set** is picked up, reducing the number to 12 again.

The play continues until the deck is depleted. At the end of the play there may be 0, 3, 6, 9, 12 or 15 cards remaining which do not form a **set**. The number of **sets** held by each player are then counted. One point is given for each **set** and added to their score. The deal than passes to the person on dealer's left and the play resumes with the deck being reshuffled. When all the players have dealt, the game ends; highest score wins. The nice part about SET® is that it can keep children and adults busy for hours.

4.3.1 History of the game SET®

The game SET® is a card game that was first created in 1974 by the geneticist, Marsha Jean Falco. Falco was studying epilepsy in German Shepherds. She began to represent the genetic information about dogs on cards using different symbols and began to look for patterns within the cards. It was then she realized that searching for patterns created a challenge and could possibly be used as a card game [31]. The game was first released to the public in 1991 and "soon became a popular game among college students" [50]. SET® has won several awards since it was first released to the public in 1991. A full list of the awards that SET® has won can be found in Appendix D.

For anyone who is interested in playing SET® the website www.setgame.com provides a daily SET® puzzle. There are also SET® local competitions as well as a national SET® tournament which, is held every year.

4.3.2 SET® in the High School Classroom

SET® is a useful game for introducing students to abstract mathematical ideas and topics. According to Hannah Gordon, Rebecca Gordon, and Elizabeth McMahon in their paper "Hands-on SET®", SET® can be used to introduce students to topics such as combinatorics, probability, finite geometry, and linear algebra [22]. Gordon, Gordon and McMahon further explain how SET® can be used for a High School class by saying: "The cards in the game of SET® provide an excellent model for finite affine geometry. The game helps students visualize the geometry, and the geometry provides insight into the game as well" [22].

This lesson should be covered in two or more class periods, depending upon the students. The main objectives for this lesson will be:

1. To learn to complete a **set** when fixing two characteristics.

2. To learn how write set cards as coordinates.

3. To learn how to complete a **set** when given two random cards or coordinates.

The NCTM Standards this lesson will cover include:

> **Grades 9-12** Standards 1a, 2a, 2d, 3, 5a, 5c, 5d, 6, 7, 8, 9, 10

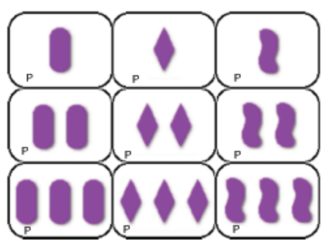
Consider for example the standard 1.a: "Understand numbers, ways of representing numbers, relationships among numbers, and number systems." In this lesson students will learn how to represent the SET® cards as a **set** of coordinates and how to convert the back and forth between the SET® cards and the points which represent the cards.

Another example of a standard that this lesson will address is 5.c., "Develop and evaluate inferences and predictions that are based on data." Here students will use inference to determine which cards are needed to compete a given **set** of cards.

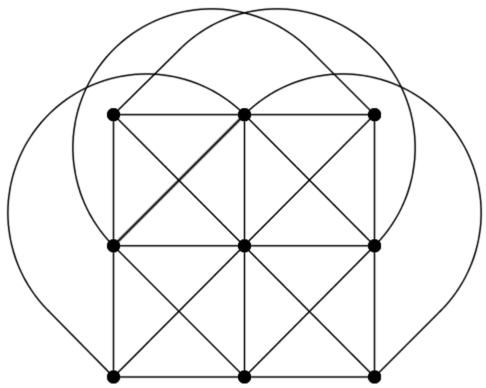
The Lesson: When introducing the game of SET® to students who are unfamiliar with the game, the teacher should start by providing the history of the game as well as how the game is typically played. Once students have spent some time learning the game, the teacher should then have the students break up into groups of three or four and allow the students to play for the remainder of the hour.

The next day the teacher should start by making connections between playing TTT and playing SET® in $AG(2,3)$. Gordon, Gordon, and McMahon also explain that "[f]or students who do not know about more abstract geometries, you [the teacher] can have them view the deck as a finite affine geometry: the cards are the points in the geometry, and the three points are on a line if those three cards form a set" [22]. Some of the students will begin to see the connection between the two games. For those students who are still struggling, the following should help students begin to see how to use affine geometry when playing SET®. Gordon, Gordon, and McMahon begin by suggesting for the teacher to "[a]sk the students to isolate two attributes, and then find nine cards that have those attributes in common" [22]. The example that Gordon, Gordon, and McMahon suggest is to have the students start off by isolating the attributes of color and shading. Gordon, Gordon, and McMahon give the example of having students isolate the attributes solid and purple [22]. By isolating the attributes of solid and purple the students should have the following cards separated:

Once the students have separated the two attributes from the rest of the deck, the teacher should ask the students, as Gordon, Gordon, and McMahon suggest, to "organize the cards in a nice way that helps them see all the **sets** that those nine cards contain" [22]. Once the students have identified all of the **sets** in the group, Gordon, Gordon, and McMahon predict the majority of students will create the following 3×3 square:



The teacher may want to note the connection to $AG(2,3)$ by providing the picture of $AG(2,3)$ where the students can see the 12 lines that are formed: "three horizontal lines, three vertical lines, three lines parallel to the main diagonal . . . and three lines parallel to the opposite diagonal" [22]. Students should be able to see the connection between the 12 lines and the collection of **sets** they created with their cards with two attributes:



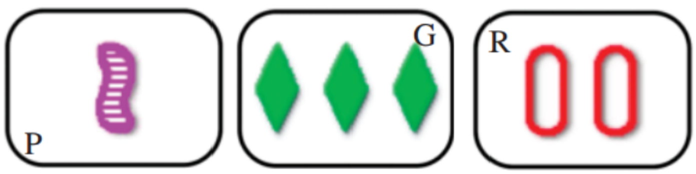
After the teacher has shown students the connection, the teacher should begin to introduce or review planar affine geometry. Gordon, Gordon, and McMahon suggest having students recall the following **set** of axioms for planar affine geometry as well as providing interpretations to the game of SET® [22].

1. Axiom 1: Given any two distinct points P and Q, there is exactly one line passing through them. (Translation: Two cards determine a unique **set**. This is what we have called the Fundamental Theorem of SET®.)
2. Axiom 2: Given any line l and a point P not on l, there is exactly one line through P parallel to l. (Translation: Given a **set** and a card not in the **set**, there's a unique parallel **set**.)
3. Axiom 3: There are at least three non-collinear points. (Translation: There are at least three cards that do not form a **set**. This axiom guarantees that there are enough points and lines to make the geometry interesting.)

The teacher can now allow students to learn how to determine what card is needed next to complete a **set** when given just two cards. The teacher should start off the class by having students give examples of their thoughts on how they will determine what card is needed to complete the **set**. Once students have discussed their strategies, the teacher should take a deck of SET® cards and deal two cards.

Once the teacher has dealt two cards the teacher should ask students to tell what is the next carded needed to form a **set**. An example of a possible scenario is provided below.

Once the students have had time to think about which card is missing, the students should come to the conclusion that the next carded needed to complete the **set** is the two red ovals, which creates the following as the solution.

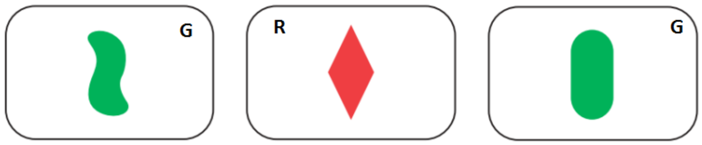


The teacher should allow students to break back up into pairs of two and have each student take turns dealing two cards and have the other student state which card completes the **set**. Through this process the students become more familiar with the process of finding which card is needed to complete the **set**.

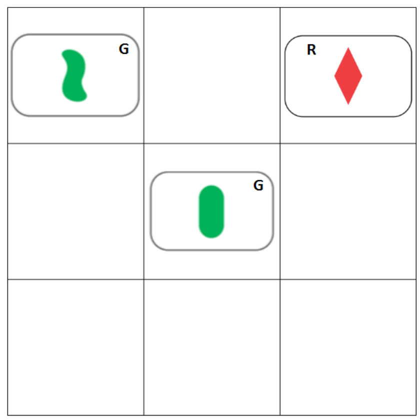
We can see how the game of SET® provides a secondary method for understanding these axioms. As an extension, the teacher should point out as stated by Gordon, Gordon, and McMahon:

> [n]ot only do any two cards define a set, any three cards that are not a set define an entire magic square (that is, a plane). The configuration is unique in the sense that if you take any three cards that aren't a set from that square, put them in the upper left and complete a new square as before, it will consist of the same nine cards [22].

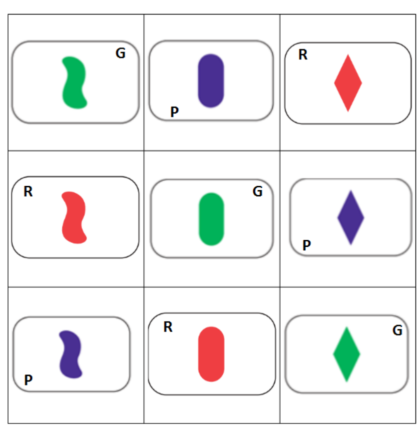
The following are three SET® cards that do not complete a **set** as provided by Llewellyn Falco [17] in his paper entitled "Mathematical Proof of the Magic Squares":



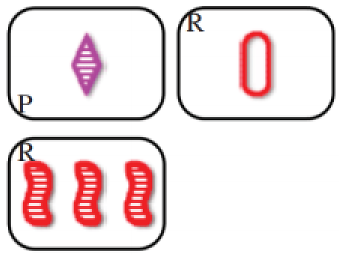
The teacher should have the students place the three drawn cards as shown down below.



The teacher should now ask students to begin work on their own, filling in the rest of the missing pieces. The first box the student should fill in is the top middle box which is a (one, purple, solid, oval). The students might also realize that the diagonal is green, which would by deduction mean that the bottom right corner is a (one, green, solid, diamond). The students should be able to begin to fill out the rest of the square so that the final result is the following picture provided by Falco [17].

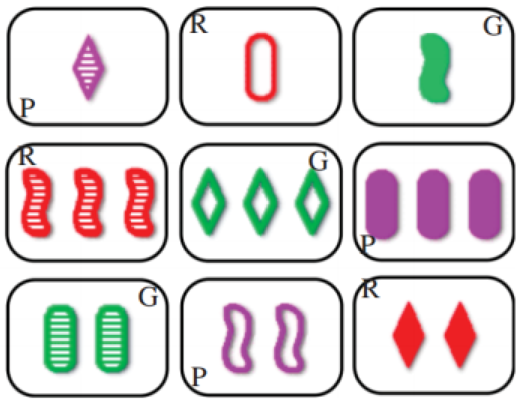


By extending the first lesson of separating all the SET® cards by two attributes to now having the students choose three cards that are not in a **set** and having the students find cards that will complete the magic square, the teacher can now suggest the following provided by Gordon, Gordon, and McMahon. They suggest to "[t]ake any three cards that are not a set and place them in the upper left corner of a rectangle" [22]. An example group can be seen below.



The teacher should have the students continue with playing with choosing three cards that do not form a **set**. The teacher can ask the students to try to complete a magic square with cards that do form a **set**. Falco notes that "[i]t will work with a set but the square becomes redundant" [17].

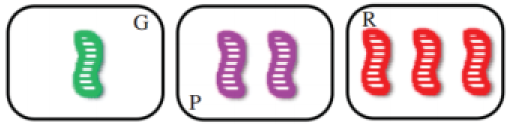
Have each group of students (with their own deck) deal themselves three random cards. If they form a **set** have them deal three new cards until they get three that are not a **set**. Gordon, Gordon, and McMahon tell the teacher to have the students "[f]ind the cards that would complete the horizontal and vertical sets, and then add the card in the center that completes a set with the two new cards. The last three cards are added so every row and column are sets (the order you complete the square doesn't matter, as you can see by inspection)" [22]. Gordon, Gordon, and McMahon provide the following picture as an example of a completed plane of cards from the original three cards that did not form a **set**.



Once the students are comfortable completing **sets** and magic squares using the cards of SET®, the teacher should discuss with students about coordinatizing (numerical labels) each SET® card, which exposes the students to linear algebra. Gordon, Gordon, and McMahon explain that "[v]iewing the cards this way will allow us to learn even more about the game, which is a wonderful way of reinforcing how useful linear algebra is [22]. The teacher needs to assign numbers 0, 1, and 2 that will correspond to any possible attributes of a SET® card. Gordon, Gordon, McMahon state "[i]t doesn't matter how you do this, but once you've chosen your numbering, you need to keep it fixed" [22]. Gordon, Gordon, and McMahon provide the following table as a possible way to number the attributes.

	Attribute		
	0	1	2
Number	3	1	2
Color	green	red	purple
Shading	empty	striped	solid
Shape	diamond	oval	squiggle

Gordon, Gordon, McMahon further provide the teacher with a picture of three cards for the students to use the above table to convert the cards to coordinates [22].



Since the cards can be thought of as points, labels for the coordinates of the ordered quadruple can be written using the convention $(Number, Color, Shade, Shape)$. Once the students have had a few minutes to work on coordinatizing, the teacher should check and see if the students got the following as their answers: $(1, 0, 1, 2)$, $(2, 2, 1, 2)$, and $(0, 1, 1, 2)$ [22]. Once students have been able to grasp the concept of looking at cards and being able to convert to an ordered quadruple, students can also be given an ordered quadruple and have to find out what card the coordinates describe. For example, the teacher can give students the following point to have them figure out what card the point is describing: $(0, 2, 0, 1)$. This point is three purple empty ovals. Having the students convert back and forth is a good activity for students to do at the beginning of class as a bell-work problem, while the teacher is taking roll and preparing to begin the lesson for the day.

Now that students have become familiar learning how to represent SET® cards as coordinates, the teacher should expand to the idea of completing a **set** of given two coordinates. For example, the teacher could write the following pair of coordinates on the board and have the students spend some time figuring out which point is needed to complete the **set** of cards.

$$(2, 2, 2, 2) \text{ and } (2, 0, 1, 0)$$

The students should begin to realize that the first coordinate is the same in both ordered quadruples which means that a 2 is also needed in the first coordinate to complete the **set**. Continuing with the same process, the students should have the following as their solution: $(2, 1, 0, 1)$. Now that the students understand the process of completing a **set** using the coordinates, the teacher can take a deck of SET® cards and walk around the room placing two cards on each student's desk. Once the teacher has given each student two cards he/she can have students take out two pieces of paper and ask the students to choose one of the two pieces of paper to write the card that will complete the **set**. Once students have written the card that will complete the **set**, the teacher should have the students write the coordinates of two cards that were handed out. Once the students have completed converting the cards to coordinates, the teacher should have the students write the those coordinates on the other sheet of paper so that the students will be able to trade with another student in the classroom. Before the teacher allows students to trade papers, the teacher should have students convert the original missing card into coordinates on the paper that the missing card was originally written on. Once the students have completed all of the conversions, the teacher should have the students trade with each other and have the students finding the coordinates which will complete the **set**. Once the students believe that they have the correct answer, the teacher should have the students check with the original owner and verify that they have the correct solution. The teacher may also find it useful to use this activity as bell-work or even as a bonus question on a test or quiz once the students have had time to practice switching back and forth between completing a **set** with the SET® cards and completing a **set** with only the coordinates.

The teacher can now have the students use what they learned about SET® coordinates to complete a magic square using coordinates to find the rest of the missing cards given three initial coordinates. The teacher should have the students recall that the ordered quadruple is written as follows $(Number, Color, Shade, Shape)$. The teacher should have the students the complete a magic square with the following coordinates: $(0, 1, 2, 0)$, $(0, 2, 0, 1)$ and $(1, 0, 1, 1)$. The teacher may find it helpful to place these coordinates on a standard 3×3 board so that it looks like the following.

	(0, 2, 0, 1)	
(0, 1, 2, 0)		(1, 0, 1, 1)

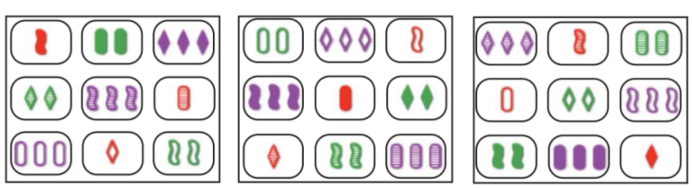
The students should follow the same logic and process as they learned when completing a magic square with cards as well as when completing a **set** for a given pair of coordinates. The students should be able to begin to solve the magic square by noticing that the easiest place to begin is the middle square of the 3×3 board, since by completing this square students will have completed one **set** and be able to solve and complete the rest of the magic square.

Once the students have continued to use logic and deduction to fill in the remainder of the magic square, they should end with the following solution:

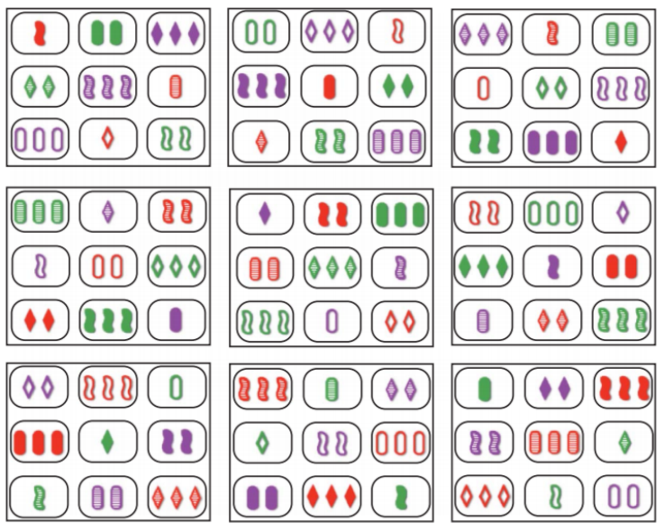
(1, 1, 2, 2)	(0, 2, 0, 1)	(2, 0, 1, 0)
(0, 1, 2, 0)	(2, 2, 0, 2)	(1, 0, 1, 1)
(2, 1, 2, 1)	(1, 2, 0, 0)	(0, 0, 1, 2)

The teacher should remind students that each **set** of coordinates correspond to a line in the affine plane and that each parallel class of coordinates also form a parallel class in $AG(2,3)$.

There are several possible extension of playing and using SET® in the classroom. After students become comfortable with magic squares in SET® as representing $AG(2,3)$, the teacher can expand the idea of using SET® in three-dimensions to represent $AG(3,3)$ by, as Gordon, Gordon, and McMahon suggest, "building a cube" [22]. The picture below is a picture of a $AG(3,3)$ provided by Gordon, Gordon, and McMahon [22].



A further extension for using SET® would be if average students or more advanced students grasp the concept of representing $AG(3,3)$ in three-dimensions and are comfortable, the teacher can expand the process even further into four-dimensions and use the entire deck to represent $AG(4,3)$ [22]. The following is an example of a representation of $AG(4,3)$ provided by Gordon, Gordon, and McMahon [22]. The teacher should ask students to find **sets** in each representation and to explain how the **set** corresponds to a line in the space. Using the deck of SET® cards to represent a $AG(4,3)$ provides a unique way to "visualize" a four-dimensional space. Advanced students could also be asked to define parallel planes and parallel spaces using the representation.



Another possible extension when using SET® and magic squares is for the teacher to explain the connection with looking only at the cards as **sets** of coordinates and discussing the connection of how error-correction codes work in cryptology [31].

Chapter 5

Conclusion

In this thesis, I have discussed the benefits of using discrete mathematics in the K-12 curriculum. I have also focused on three topics within discrete mathematics and, within those topics, picked two lessons that a mathematics teacher could use in conjunction with his/her current curriculum. Each of the problems that this thesis focused on offers students at any level of academic education the ability to study and learn higher levels of abstract mathematics as well as to see a direct application of mathematics in the real world. Discrete mathematics offers a unique opportunity for a break to students from their study of continuous mathematics, which prepares them to take Calculus. Since many students do not plan on taking Calculus, allowing them to use and learn discrete mathematics in the K-12 curriculum provide an opportunity for them to pick up those problem solving skills in an alternative format. Discrete mathematics also offers students who are struggling or who have lost interest in mathematics an opportunity to reengage with the material and to build mathematical confidence. Another benefit of using and incorporating discrete mathematics into the curriculum is the types of problems addressed by discrete mathematics are relatively easy for students to understand, whether it is having to decipher a code, finding the most efficient and cost effective path, coloring a map with the fewest colors, or playing mathematical games. Each of these questions is accessible for students of all ages and all academic backgrounds.

By using discrete mathematics in the K-12 classroom the teacher is able to expose students to higher level mathematics at a younger age and allow them to see the usefulness of mathematics in the real world. Also, using discrete mathematics in the K-12 classroom provides students with current mathematics topics and more relevant topics in an ever increasing computer culture. Students from birth are exposed to computers and by using discrete mathematics in the classroom, they are able to understand the mathematics behind the computers that they use everyday.

In each of the topics covered in this thesis, students should be able to grasp the necessary concepts to solve the problems regardless of previous mathematical background. The Caesar cipher is a good introduction for students at the elementary level to the use of encoding and decoding messages. Here students are able to learn both the history and the importance of keeping information hidden. Students are also exposed to modern cryptology when learning about the traditional substitution cipher. Here students at the junior high level will be able to build upon the material learned about the Caesar cipher. While using the substitution cipher, students will be able to learn about modular arithmetic and how to use letter frequencies like Al-Kindi used when he tried to decipher messages.

Topics in Graph Theory are also useful when exposing students in the K-12 classroom to topics of discrete mathematics. Two of those topics are the Chinese Postman Problem (CPP) and the other is the Four-Color Theorem. The CPP allows students to be able to find an optimal route for a postman to travel every edge of a graph with minimizing the number of times he/she has to cross the same edge again. Here students are able to find the optimal solution if there are an even number of vertices and how to create a path if there are only an odd number of vertices. They are also able to see how the CPP is used in garbage pickup and snow removal for cities and towns.

The Four-Color Theorem exposes students to finding the minimum number of colors to use to color objects, whether coloring a map of the United States of America, or coloring a picture of a pie. The Four-Color Theorem allows students to be able to see the applications of the Four-Color Theorem in areas such as scheduling, transportation of chemicals, and Global Systems for Mobil Communications (GSM).

Teachers are able to use mathematical games such as Tic-Tac-Toe (TTT) and SET® when introducing students to abstract mathematics at the K-12 level. Many students know how to play games such as TTT and SET®, but many do not know the unique properties each game employs with geometry. TTT is a game that students learn at an early age and for many students TTT can become boring rather quickly once students are able to find strategies that create tie games. By taking a typical TTT game and adding another level of difficulty by playing in $AG(2,3)$, students can be challenged to think critically and more abstractly, allowing students to build cognitive reasoning skills.

SET® also has unique mathematical properties that allow students to use mathematical reasoning to play. The game itself exposes students to advanced mathematics such as the branch of mathematics know as set theory, which is the study of the **sets**. **Sets** are a collection of finite or infinite objects. However, by studying the mathematics behind SET® with various restrictions, students are required to make **sets** and find **sets** when given three cards that do not form a **set**. By playing SET® with various restrictions and studying the mathematics behind those variations, students are able to develop the necessary problem solving skills that are needed when solving and studying advanced mathematics.

For future study I would like to pair with local schools to implement these three areas in the mathematics curriculum. I am interested in seeing how students score on their standardized test both before and after they are exposed to topics in discrete mathematics. I am also interested in looking at students pre and post tests when these three areas have been implemented. I would also be interested in seeing if students' motivation for learning mathematics is increased when learning discrete mathematics.

Other areas that I would like to study, include RSA encryption into the curriculum as well. RSA encryption is used in modern computers to keep information secure. I believe that incorporating RSA encryption into the cryptology section will help students understand how far cryptology has come and the importance needed in developing even more complex encryption algorithm as technology continues to develop. I would also like to study and incorporate different games to be played on affine planes such as chess and checkers. I am interested in what those games would look like and how students would approach learning to play games such as these on different boards.

Writing this thesis has been a unique experience and an enjoyment. I have been able to reflect on my own time as a student as well as approach writing this thesis from a teacher's perspective. As I reflected on my time as a student, I remembered asking those questions of "Why am studying this?", "When am I ever going to use this?" and "How is this used in the real world?" It was not until I took classes in Discrete mathematics, Cryptology, and Abstract Algebra that I began to understand why studying traditional mathematics was necessary, and I realized that many students will never take classes past college algebra. As I took those advanced classes I began to see how some of the topics that I was studying could be used in the K-12 classroom. As a teacher, I have been able to see topics within cryptology and Graph Theory answers those questions of "why" and "when". I am interested in continuing my research within this area of mathematics education, and I hope that these lesson plans can be used and implemented in the K-12 classrooms to help inspire students to study mathematics.

Guide to the Appendices

- Appendix A contains a list of all of the National Council of Teachers in Mathematics (NCTM) standards for each of the grades K-12.
- Appendix B contains possible worksheets to be used for each of the two problems, along with a Caesar cipher wheel to be used with the Caesar cipher. This Appendix also contains a letter frequency chart to be used with the substitution cipher.
- Appendix C contains possible worksheets to be used for teaching the Chinese Postman Problem and the Four-Color Problem.
- Appendix D contains a paper of orthogonal Latin squares of order 2 and order 3. This paper can be used when playing Tic-Tac-Toe on $AG(2,3)$ and $AG(2,4)$. Appendix D also contains a worksheet to help students learn about Latin squares and orthogonal Latin squares. This Appendix also contains a list of all of the awards that the game SET® has won since it was first released to the public in 1991.

Appendix A

NCTM Math Standards

These standards are taken from the book *Principles and Standards for School Mathematics* [13], which is written and produced by the National Council of Teachers of Mathematics (NCTM).

A.1 NCTM Pre-K-2 Standards

1. Number and Operations
   (a) Understanding numbers, ways of representing numbers, relationships among numbers, and number systems
   (b) Understand meanings of operations and how they relate to one another
   (c) compute fluently and make reasonable estimates
2. Algebra
   (a) Understand patterns, relations, and functions
   (b) Represent and analyze mathematical situations and structures using algebraic symbols

   (c) Use mathematical models to represent and understand quantitative relationships

   (d) Analyze change in various contexts

3. Geometry

   (a) Analyze characteristics and properties of two-and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

   (b) Specify locations and describe spatial relationships using coordinate geometry and other representational systems

   (c) apply transformations and use symmetry to analyze mathematical situations

   (d) Use visualizations, spatial reasoning, and geometric modeling to solve problems

4. Measurement

   (a) Understand measurable attributes of objects and the units, systems, and processes

   (b) Apply appropriate techniques, tools, and formulas to determine measurements

5. Data Analysis and Probability

   (a) Formulate questions that can be addressed with data and collect, organize and display relevant data to answer them.

   (b) Select and use appropriate statistical methods to analyze data

   (c) Develop and evaluate inferences and predictions that are based on data

   (d) Understanding and apply basic concepts of probability

6. Problem Solving

   (a) Build new mathematical knowledge through problem solving

   (b) Solve problems that arise in mathematics and in other contexts

   (c) Apply and adapt a variety of appropriate strategies to solve problems

   (d) Monitor and reflect on the process of mathematical problem solving

7. Reasoning and Proof
    (a) Recognize reasoning and proof as fundamental aspects of mathematics
    (b) Make and investigate mathematical conjectures
    (c) Develop and evaluate mathematical arguments and proofs
    (d) Select and use various types of reasoning and methods of proof.
8. Communication
    (a) Organize and consolidate their mathematical thinking through communication
    (b) Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
    (c) Analyze and evaluate the mathematical thinking and strategies of others
    (d) Use the language of mathematics to express mathematical ideas precisely
9. Connections
    (a) Recognize and use connections among mathematical ideas
    (b) Understanding how mathematical ideas interconnect and build on one another to produce a coherent whole
    (c) Recognize and apply mathematics in contexts outside of mathematics
10. Representation
    (a) Create and use representations to organize, record, and communicate mathematical ideas
    (b) Select, apply, and translate among mathematical representations to solve problems
    (c) Use representations to model and interpret physical, social, and mathematical phenomena.

A.2 NCTM Grades 3-5 Standards

1. Number and Operations
   (a) Understand numbers, ways of representing numbers, relationships among numbers, and number systems
   (b) Understand meanings of operations and how they relate to one another
   (c) Compute fluently and make reasonable estimates
2. Algebra
   (a) Understand patterns, relations, and functions
   (b) Represent and analyze mathematical situations and structures using algebraic symbols
   (c) Use mathematical models to represent and understand quantitative relationships
   (d) Analyze change in various contexts
3. Geometry
   (a) Analyze characteristics and properties of two-and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
   (b) Specify locations and describe spatial relationships using coordinate geometry and other representational systems
   (c) Apply transformations and use symmetry to analyze mathematical situations
   (d) Use visualizations, spatial reasoning, and geometric modeling to solve problems
4. Measurement
   (a) Understand measurable attributes of objects and the units, systems, and processes of measurement
   (b) Apply appropriate techniques, tools, and formulas to determine measurements

5. Data Analysis and Probability
   (a) Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them.
   (b) Select and use appropriate statistical methods to analyze data
   (c) Develop and evaluate inferences and predictions that are based on data
   (d) Understand and apply basic concepts of probability
6. Problem Solving
   (a) Build new mathematical knowledge through problem solving
   (b) Solve problems that arise in mathematics and in other contexts
   (c) Apply and adapt a variety of appropriate strategies to solve problems
   (d) Monitor and reflect on the process of mathematical problem solving
7. Reasoning and Proof
   (a) Recognize reasoning and proof as fundamental aspects of mathematics
   (b) Make and investigate mathematical conjectures
   (c) Develop and evaluate mathematical arguments and proofs
   (d) Select and use various types of reasoning and methods of proof
8. Communication
   (a) Organize and consolidate their mathematical thinking through communication
   (b) Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
   (c) Analyze and evaluate the mathematical thinking and strategies of others
   (d) Use the language of mathematics to express mathematical ideas precisely
9. Connections

(a) Recognize and use connections among mathematical ideas

(b) Understand how mathematical ideas interconnect and build on one another to produce a coherent whole

(c) Recognize and apply mathematics in contexts outside of mathematics

10. Representation

(a) Create and use representations to organize, record, and communicate mathematical ideas

(b) Select, apply, and translate among mathematical representations to solve problems

(c) Use representations to model and interpret physical, social, and mathematical phenomena

A.3 NCTM Grades 6-8 Standards

1. Number and Operations
   (a) Understand numbers, ways of representing numbers, relationships among numbers, and number systems
   (b) Understanding meanings of operations and how they relate to one another
   (c) Compute fluently and make reasonable estimates
2. Algebra
   (a) Understand patterns, relations, and functions
   (b) Represent and analyze mathematical situations and structures using algebraic symbols
   (c) Use mathematical models to represent and understand quantitative relationships
   (d) Analyze change in various contexts
3. Geometry
   (a) Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
   (b) Specify locations and describe spatial relationships using coordinate geometry and other representational systems
   (c) Apply transformations and use symmetry to analyze mathematical situations
   (d) Use visualizations, spatial reasoning, and geometric modeling to solve problems
4. Measurement
   (a) Understand measurable attributes of objects and the units, systems, and processes of measurement
   (b) Apply appropriate techniques, tools, and formulas to determine measurements

5. Data Analysis and Probability
    (a) Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them
    (b) Select and use appropriate statistical methods to analyze data
    (c) Develop and evaluate inferences and predictions that are based on data
    (d) Understand and apply basic concepts of probability
6. Problem Solving
    (a) Build new mathematical knowledge through problem solving
    (b) Solve problems that arise in mathematics and in other contexts
    (c) Apply and adapt a variety of appropriate strategies to solve problems
    (d) Monitor and reflect on the process of mathematical problem solving
7. Reasoning and Proof
    (a) Recognize reasoning and proof as fundamental aspects of mathematics
    (b) Make and investigate mathematical conjectures
    (c) Develop and evaluate mathematical arguments and proofs
    (d) Select and use various types of reasoning and methods of proof
8. Communication
    (a) Organize and consolidate their mathematical thinking through communication
    (b) Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
    (c) Analyze and evaluate the mathematical thinking and strategies of others
    (d) Use the language of mathematics to express mathematical ideas precisely
9. Connections

   (a) Recognize and use connections among mathematical ideas
   (b) Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
   (c) Recognize and apply mathematics in contexts outside of mathematics

10. Representations
    (a) Create and use representations to organize, record, and communicate mathematical ideas
    (b) Select, apply, and translate among mathematical representations to solve problems
    (c) Use representations to model and interpret physical, social, and mathematical phenomena

A.4 NCTM Grades 9-12 Standards

1. Number and Operations
   (a) Understand numbers, ways of representing numbers, relationships among numbers, and number systems
   (b) Understand meanings of operations and how they relate to one another
   (c) Compute fluently and make reasonable estimates
2. Algebra
   (a) Understand patterns, relations, and functions
   (b) Represent and Analyze mathematical situations and structures using algebraic symbols
   (c) Use mathematical models to represent and understand quantitative relationships
   (d) Analyze change in various contexts
3. Geometry
   (a) Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
   (b) Specify locations and describe spatial relationships using coordinate geometry and other representational systems
   (c) Apply transformations and use symmetry to analyze mathematical situations
   (d) Use visualization, spatial reasoning, and geometric modeling to solve problems
4. Measurement
   (a) Understand measurable attributes of objects and the units, systems, and processes of measurement
   (b) Apply appropriate techniques, tools, and formulas to determine measurement
5. Data Analysis and Probability

(a) Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them

(b) Select and use appropriate statistical methods to analyze data

(c) Develop and evaluate inferences and predictions that are based on data

(d) Understand and apply basic concepts of probability

6. Problem Solving

   (a) Build new mathematical knowledge through problem solving

   (b) Solve problems that arise in mathematics and in other contexts

   (c) Apply and adapt a variety of appropriate strategies to solve problems

   (d) Monitor and reflect on the process of mathematical problem solving

7. Reasoning and Proof

   (a) Recognize reasoning and proof as fundamental aspects of mathematics

   (b) Make and investigate mathematical conjectures

   (c) Develop and evaluate mathematical arguments and proofs

   (d) Select and use various types of reasoning and methods of proof

8. Communication

   (a) Organize and consolidate their mathematical thinking through communication

   (b) Communicate their mathematical thinking coherently and clearly to peers, teachers, and others

   (c) Analyze and evaluate the mathematical thinking and strategies of others

   (d) Use the language of mathematics to express mathematical ideas precisely

9. Connections

   (a) Recognize and use connections among mathematical ideas

(b) Understand how mathematical ideas interconnect and build on one another to produce a coherent whole

(c) Recognize and apply mathematics in contexts outside of mathematics

10. Representation

(a) Create and use representations to organize, record, and communicate mathematical ideas

(b) Select, apply, and translate among mathematical representations to solve problems

(c) Use representations to model and interpret physical, social, and mathematical phenomena

Appendix B

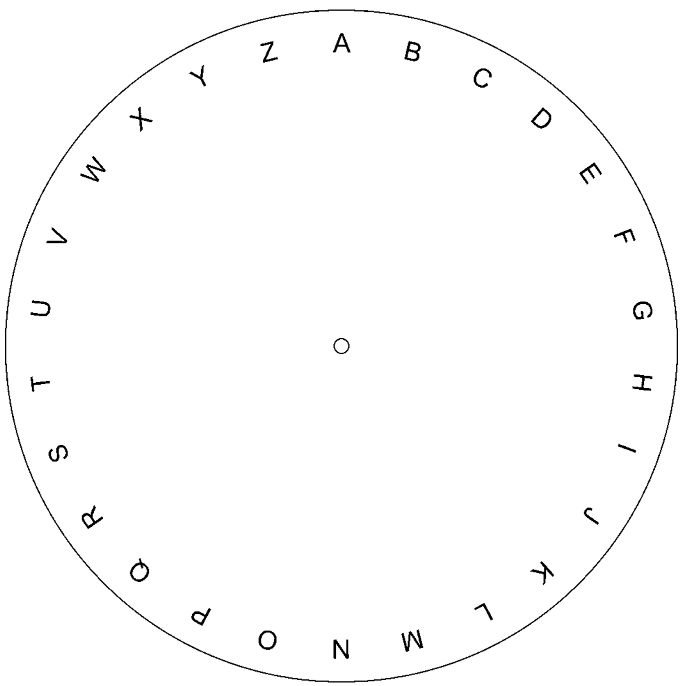
Cryptology Support Documents

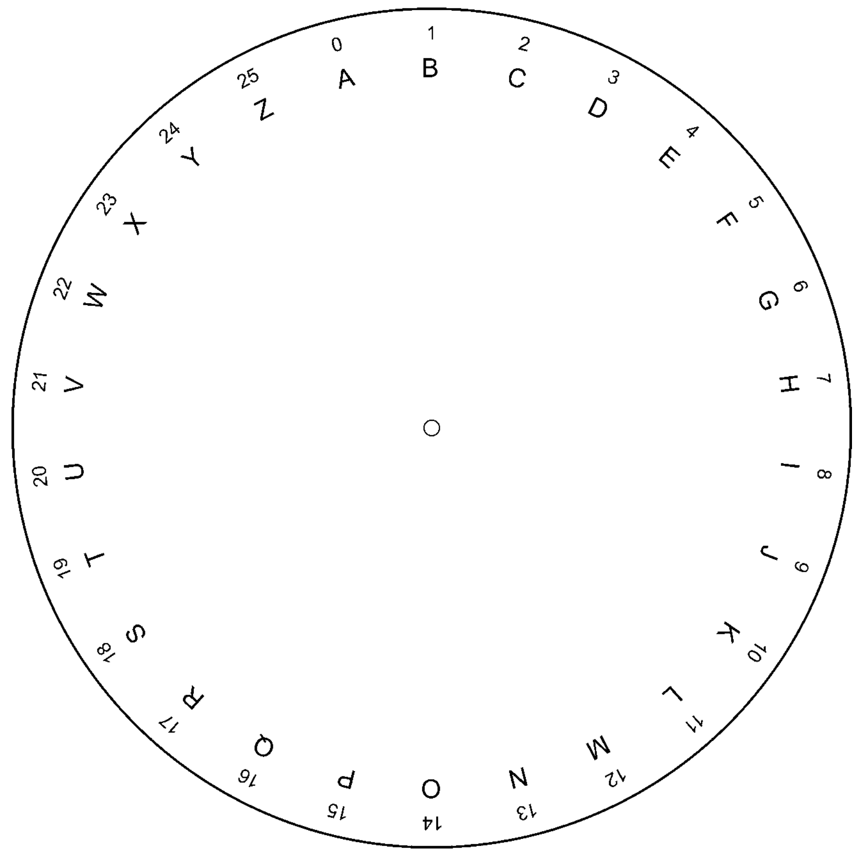
The following are support documents for use when covering Cryptography. The first document and second documents are provided by the NCTM Illuminations. The first document is a Caesar Shifter. The second document is a worksheet which is a worksheet that covers the history of the Casar cipher [27]. The third document is a Caesar cipher worksheet written by Lance Bryant and JoAnn Ward out of Purdue University [5]. The fourth document is a letter frequency chart, from the NCTM Illuminations, that can be used when covering the substitution cipher [27]. The last document is from the workbook that accompanies *The Cryptoclub* written by Janet Beissinger and Vera Pless [4].

Caesar Shifter

To use the Caesar substitution cipher, create a Caesar shifter as follows:

1. Remove the circles at the bottom of this sheet.
2. Place the smaller circle over the larger circle, and place a paper fastener through the center of both circles.
3. To encode a message, rotate the smaller circle the appropriate number of units. Then, replace plaintext letters from the smaller circle with ciphertext letters from the larger circle. (To decode a message, do the opposite.)





Caesar Cipher

NAME ____________________________

1. Who was Julius Caesar, and where was he from? During what period of time did he live?

2. The Caesar cipher is based on shifting the letters in the alphabet a certain number of spaces. How many different shifts are possible? Explain your reasoning.

3. Why do you think this method was used by Julius Caesar? At his time in history, how effective do you think this method was for sending secret messages?

4. Would this method provide adequate security today, or would a code using this method be easily broken? Explain your opinion.

5. Explain why the number of total shifts possible from Question 2 is a weakness of the Caesar cipher.

CRYPTOGRAPHY WORKSHEET

Name: ______________________________ Class: ____________

Encode the following messages.

(1) Caesar cipher with shift $+3$

hello tom

(2) Caesar cipher with shift $+12$

klondike nuggets

Decode the following messages.

(3) Caesar cipher with shift $+5$

ltytufwnx

(4) Caesar cipher with shift $+21 = -5$

adiyevhznwjiy

(5) Caesar cipher with shift $+24 = -2$

ncwrmlkyllgle

Letter Frequencies

This table shows the percent frequency of each letter in the English alphabet.

Letter	Frequency (%)
A	7.3
B	0.9
C	3.0
D	4.4
E	13.0
F	2.8
G	1.6
H	3.5
I	7.4
J	0.2
K	0.3
L	3.5
M	2.5

Letter	Frequency (%)
N	7.8
O	7.4
P	2.7
Q	0.3
R	7.7
S	6.3
T	9.3
U	2.7
V	1.3
W	1.6
X	0.5
Y	1.9
Z	1.0

Name ____ Date ______________

Chapter 5: Letter Frequencies
(Text page 37)

CLASS ACTIVITY: Finding Relative Frequencies of Letters in English

Part 1. Collecting data from a small sample.

a. Choose about 100 English letters from a newspaper or other English text. (Note: If you are working without a class, choose a larger sample—around 500 letters. Then skip Parts 1 and 2.)
b. Work with your group to count the **A**s, **B**s, etc., in your sample.
c. Enter your data in the table below.

Letter Frequencies for Your Sample

Letter	Frequency
A	
B	
C	
D	
E	
F	
G	
H	
I	
J	
K	
L	
M	

Letter	Frequency
N	
O	
P	
Q	
R	
S	
T	
U	
V	
W	
X	
Y	
Z	

Part 2. Combining data to make a larger sample.

a. Record your data from Part 1 on your class's Class Letter Frequencies table. (Your teacher will provide this table on the board, overhead, or chart paper.)
b. Your teacher will assign your group a few rows to add. Enter your sums in the group table.

Name ________________________________ Date ____________

(Text pages 37–38)

Part 3. Computing relative frequencies.

Enter your class's combined data from the "Total for All Groups" column of Part 2 into the "Frequency" column. Then compute the relative frequencies.

		Relative Frequency		
Letter	Frequency	Fraction	Decimal (to 3 places)	Percent (%) (to nearest tenth)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				
K				
L				
M				
N				
O				
P				
Q				
R				
S				
T				
U				
V				
W				
X				
Y				
Z				
Total				

Name ______________________________ **Date** ____________

(Text page 38)

1. a. What percent of the letters in the class sample were the letter **T**? ____%

   b. About how many **T**s would you expect in a sample of 100 letters? _____

   c. If your sample was about 100 letters, was your answer to 1b close to the number of **T**s you found in your sample? ____________________

2. a. What percent of the letters in the class sample were the letter **E**? ____%

   b. About how many **E**s would you expect in a sample size of 100? _____

   c. About how many **E**s would you expect in a sample of 1000 letters? _____

3. Arrange the letters in your class table in order, from most common to least common.

most common

Letter	Relative frequency %

Letter	Relative frequency %

least common

4. The table on Page 39 of the text shows frequencies of letters in English computed using a sample of about 100,000 letters. How is your class data the same as the data in that table? How is it different? Why might it be different?

____

____

____

____

Name ____ Date______________

Chapter 6: Breaking Substitution Ciphers
(Text page 49)

1. Use frequency analysis to decrypt Jenny's message, which is shown on the following page.

   a. Record the number of occurrences (frequency) of each letter in her message. Then compute the relative frequencies.

Letter Frequencies for Jenny's Message

		Relative Frequency		
Letter	Frequency	Fraction	Decimal (to 3 places)	Percent (%) (to nearest tenth)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				
K				
L				
M				
N				
O				
P				
Q				
R				
S				
T				
U				
V				
W				
X				
Y				
Z				
Total				

Name ____ Date ______________

(Text page 49)

b. Arrange letters in order from the most common to the least common.

In Message

Letter	Rel. Freq. (%)

In English

Letter	Rel. Freq. (%)
e	12.7
t	9.1
a	8.2
o	7.5
i	7.0
n	6.7
s	6.3
h	6.1
r	6.0
d	4.3
l	4.0
c	2.8
u	2.8
m	2.4
w	2.4
f	2.2
g	2.0
y	2.0
p	1.9
b	1.5
v	1.0
k	0.8
j	0.2
q	0.1
x	0.1
z	0.1

c. Now decrypt Jenny's message, using the frequencies to help you guess the correct substitutions. Record your substitutions in the Substitution Table below the message. Tip: Use pencil!

Jenny's Message

[1] Y XTNDQ DNQYS EFNFYSU

[2] JKLM NUUSGUPTQ FXTL JYII

[3] WYHT NJNL VDTT PYDPGE

[4] FYPCTFE FS FXT VYDEF

[5] FJTUFL-VYHT ATSAIT JXS

[6] PNII YU. YF ESGUQE IYCT

[7] VGU. ITF'E NII PNII NUQ

[8] WS FSWTFXTD.

Substitution Table

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

Name ____ **Date** ______________

(Text page 49)

2. Here is another message to decrypt using frequency analysis. The relative frequencies have been computed for you. Record your substitutions in the Substitution Table below the message. Tip: Use pencil!

In Message 2

Letter	Rel. Freq. (%)
D	11.4
G	9.8
Q	8.3
T	7.8
C	6.7
K	6.7
E	6.2
L	5.7
N	5.7
S	5.2
I	3.6
U	3.6
J	3.1
M	2.6
Y	2.6
A	1.6
B	1.6
H	1.6
R	1.6
W	1.6
O	1.0
Z	1.0
V	0.5
X	0.5
F	0.0
P	0.0

In English

Letter	Rel. Freq. (%)
e	12.7
t	9.1
a	8.2
o	7.5
i	7.0
n	6.7
s	6.3
h	6.1
r	6.0
d	4.3
l	4.0
c	2.8
u	2.8
m	2.4
w	2.4
f	2.2
g	2.0
y	2.0
p	1.9
b	1.5
v	1.0
k	0.8
j	0.2
q	0.1
x	0.1
z	0.1

Message 2

[1] BQGKNJG SDKT CDQ MGVLQETD

[2] BQGKNSLK G CGKNSLJD KDW

[3] SCEQT MLQ CES REQTCNGY.

[4] UKMLQTUKGTDIY, ET CGN G

[5] SEZD MLUQTDDK ALIIGQ GKN

[6] TCD RLY CGN G SEZD SEXTDDK

[7] KDAH. CD NUTEMUIIY WQLTD CDQ,

[8] "NDGQ BQGJJY, TCGKHS CDGOS.

[9] E'N WQETD JLQD RUT E'J GII

[10] ACLHDN UO."

Substitution Table

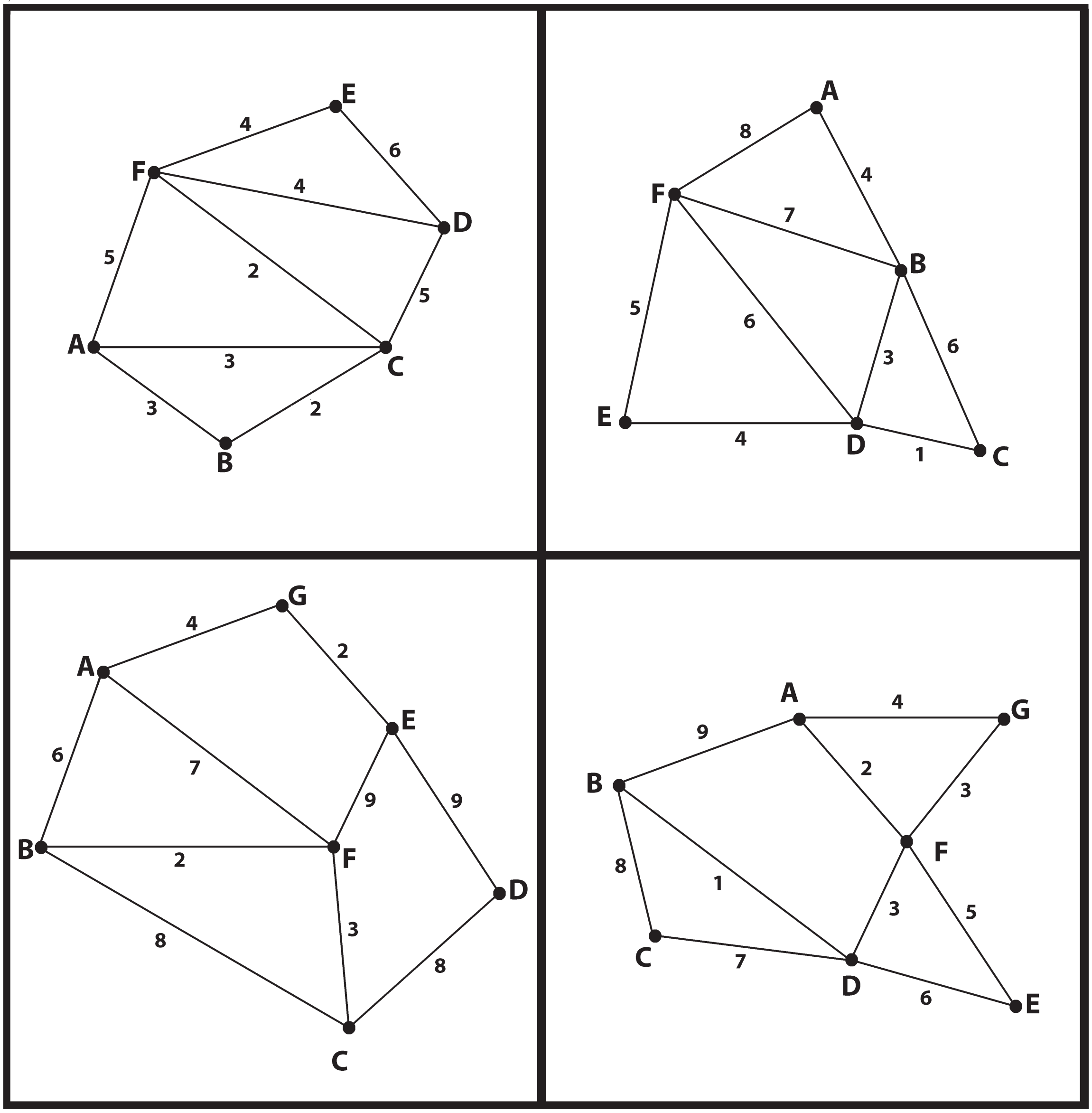
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

Appendix C

Graph Theory Support Documents

The following worksheets are support documents for Graph theory. The first set of documents are from William Emery that are from his Standards Units entitled “ Moving from Eulerian graphs to the route inspection (Chinese postman) problem” [16]. The second worksheet is from Boardman, Clough, and Evans chapter on the Chinese postman [37]. The third set of worksheets is from Terry Kawas and the website www.mathwire.com [29].

O1 Card set A – *Graphs*



O1 Card set B – *Properties*

This is an Eulerian graph because… The odd vertices are:	This is a semi-Eulerian graph because… The odd vertices are:
This is a non-Eulerian graph because… The odd vertices are:	This is a semi-Eulerian graph because… The odd vertices are:

O1 Card set C – *Completing the route*

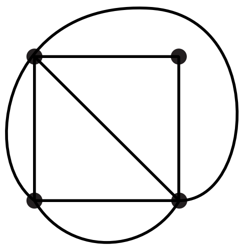
Extra edge(s) are: Possible route: Total length of route:	Extra edge(s) are: Possible route: Total length of route:
Extra edge(s) are: Possible route: Total length of route:	Extra edge(s) are: Possible route: Total length of route:

Test yourself	What to review

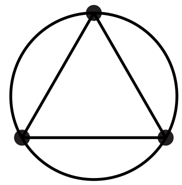
1 Which of the following networks is traversable?

Section 3.2

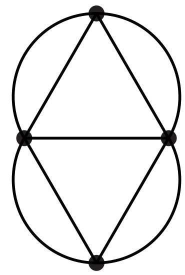
(a)



(b)



(c)



2 Find the number of ways of pairing:

Section 3.3

(a) 8 odd vertices,

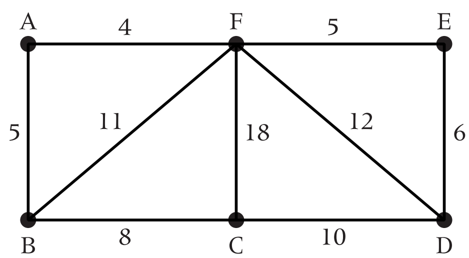
(b) 12 odd vertices,

(c) 20 odd vertices.

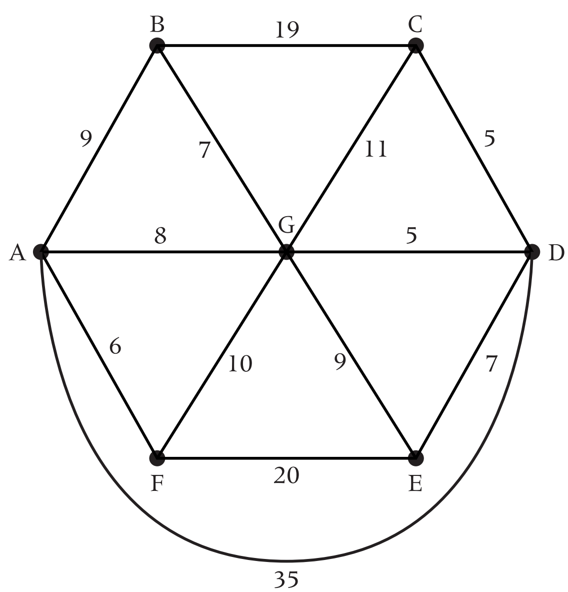
3 List the ways of pairing the odd vertices in the following networks. For each pairing find the minimum connector. Find the length of an optimal Chinese postman route. Write down one possible route.

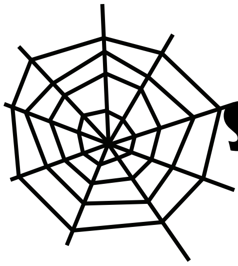
Sections 3.3, 3.4, 3.5

(a)



(b)





Spider Web Map Coloring Games

Spider Web Map Coloring Game—1

Players: 3-4

Materials:
- Spider Web Map Coloring Board—1
- 3-4 different colored markers
- 1 die (labeled 1-1-1-2-2-Lose Turn)

Directions:
- First player rolls the die and colors that many regions anywhere on the spider web following rules of map coloring:
  - Regions that share only a point may be the same color.
  - Regions that share part or all of a side may not be the same color.
- Second player rolls the die and colors that many regions anywhere on the spider web following rules of map coloring.
- Play continues until no player can color another space.

Scoring:
- Players score 1 point for each region they colored.
- Player with the highest score wins.

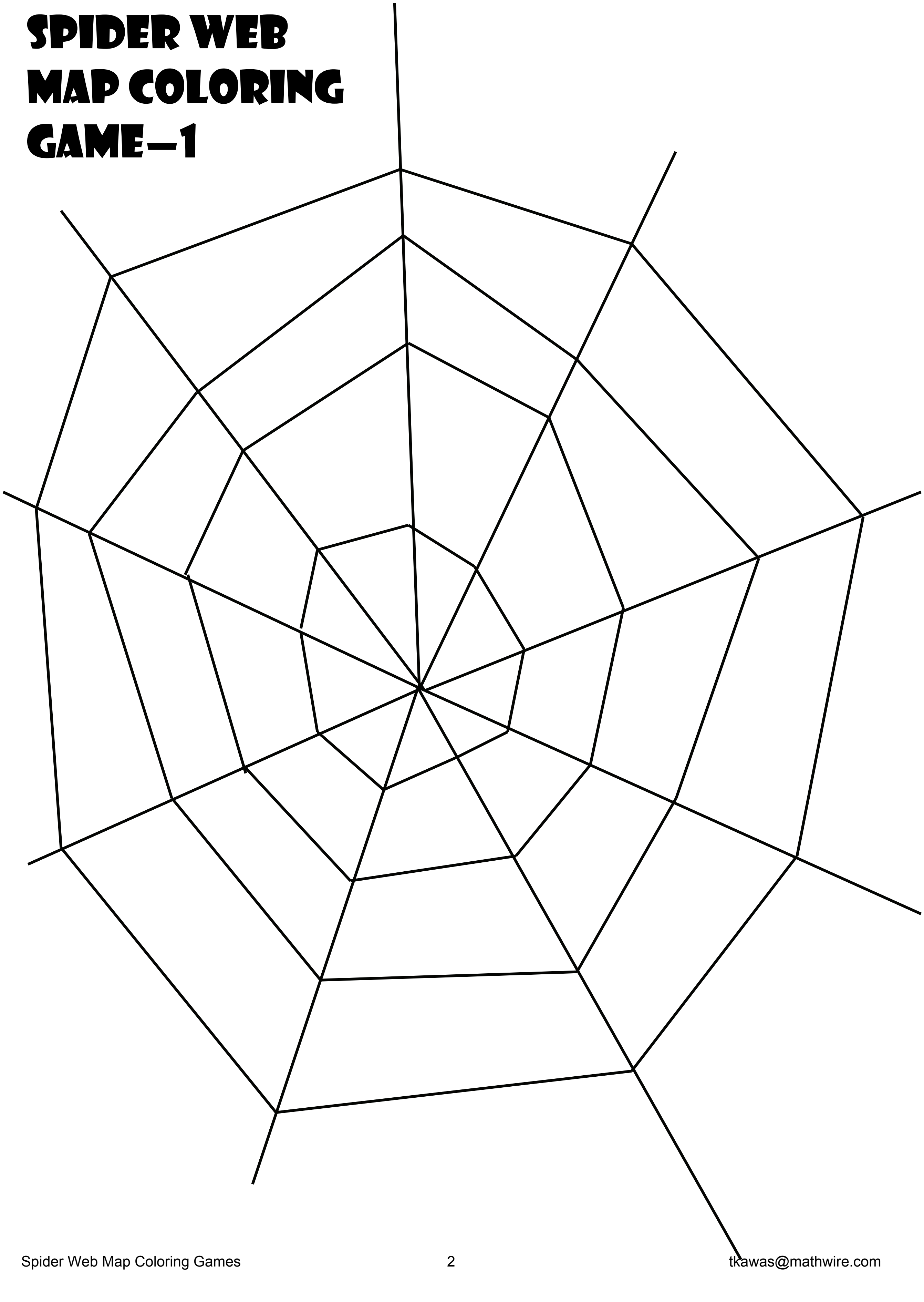
Spider Web Map Coloring Game—2

Players: 2-4

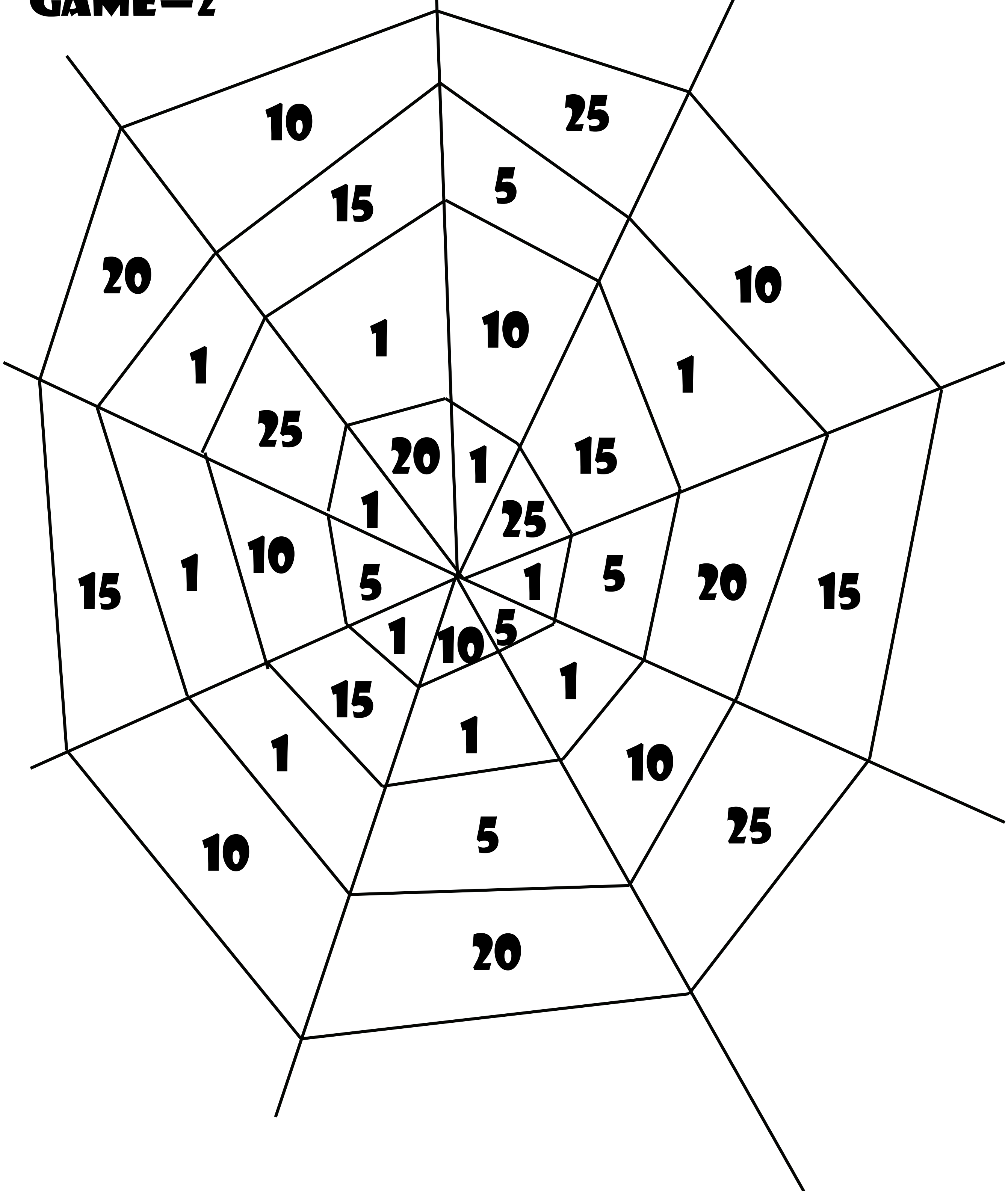
Materials:
- Spider Web Map Coloring Board—2
- 3-4 different colored markers
- 1 die (labeled 1-1-1-2-2-Lose Turn)

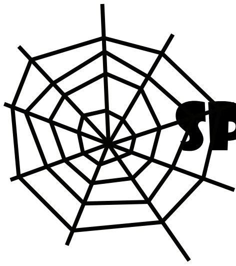
Directions:
- Play using the Spider Web Map Coloring Game –2 board.
- Players roll and color regions following map coloring rules.
- Players score the number of points assigned to each region.
- Player with the highest score wins the game.

SPIDER WEB MAP COLORING GAME–1



SPIDER WEB MAP COLORING GAME—2





SPIDER WEB MAP COLORING GAME—2

Recording Sheet

Player:		
Pts.	**Tally**	**Total**
25		
20		
15		
10		
5		
1		
GAME POINTS:		

Player:		
Pts.	**Tally**	**Total**
25		
20		
15		
10		
5		
1		
GAME POINTS:		

Player:		
Pts.	**Tally**	**Total**
25		
20		
15		
10		
5		
1		
GAME POINTS:		

Player:		
Pts.	**Tally**	**Total**
25		
20		
15		
10		
5		
1		
GAME POINTS:		

Appendix D

Game Theory Support Documents

The following are support documents for Game theory. The first document is a worksheet created by Yul Inn from his Fun Math Club website [28]. The next document is a document of Latin Squares of order 3 and 4, which should be used when playing Tic-Tac-Toe in $AG(2,3)$ and $AG(2,4)$. The Latin squares come from the paper written by Maureen T. Carroll and Steven T. Dougherty [9]. The last material that is included is a list of the awards which SET® has won since it was first released to the public in 1991.

LATIN SQUARES OF ORDER 3 AND 4

0	1	2
2	0	1
1	2	0

0	1	2
1	2	0
2	0	1

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

Latin Squares

Latin Square Puzzles

Fill in the cells below using the numbers 1, 2, and 3 to make Latin Squares:

		2
	1	
3		

1	2	
		1

Fill in the cells below using the numbers 1, 2, 3, and 4 to make Latin Squares.

3			1
		4	
1			

3			
	1		
			2
		4	

Are these Squares Orthogonal?

1

3	1	2
1	2	3
2	3	1

2

3	2	1
2	1	3
1	3	2

1 and 2 combined

Orthogonal: Yes or No?

1

1	4	3	2
2	3	4	1
3	2	1	4
4	1	2	3

2

1	3	2	4
2	4	1	3
3	1	4	2
4	2	3	1

3

1	2	4	3
2	1	3	4
3	4	2	1
4	3	1	2

1 and 2 combined

Orthogonal: Yes or No?

1 and 3 combined

Yes or No?

2 and 3 combined

Yes or No?

SET® Awards

- **MENSA**, the high IQ society, in a national competition chose SET® as one of the top five games for 1991.
- **OMNI *Magazine*** chose SET® as a top game for 1991.
- ***The Detroit News*** gave SET® “4 Stars” (its highest ratings) in its 1993 game survey.
- ***The Canadian Toy Testing Council*** awarded SET® its highest ratings of “3 Stars” in 1992.
- ***Games Magazine*** chose SET® as one of the top games of 1992, 1993, 1994 and 1995.
- ***The Consumers Association of Quebec*** awarded SET® “5 Stars” (their highest rating) in 1992. In addition SET® received their special AWARD of EXCELLENCE.
- ***Dr. Toy’s 100 Best Children’s Products for 1996*** recognizing high quality educational, creative and unique products.
- ***Deutsche Speili Preis 1995*** top ten games in Germany.
- ***Dr. Toy’s 10 Best Games for 1996*** games that offer something ‘extra’ for optimal learning and fun.
- ***ASTRA Top Toy Pic*** in 1996.
- ***Parents Choice Award 1997*** best products for children.
- ***Parents Magazine*** 1998.
- ***Parent’s Council Award*** in 1999.
- ***Top Ten Games- Wizards of the Coast*** in 2000.
- ***Teachers choice Learning Award*** in 2001.
- ***Educational Clearinghouse A+ Award*** in 2001.

- ***Bernie's Major Fun Award*** in 2002.
- ***NSSEA-Top New Product*** in 2002.
- ***ASTRA Hot Toys*** in 2004.
- ***Parents' Choice Best 25 games of the past 25 years*** in 2004.
- ***Top 100 Games of 2005 Games Quarterly*** in 2005.
- **textitTDmonthly Top-10 Most Wanted Card Games** in 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013 and 2014.
- ***Creative Child's Preferred Choice Award*** in 2007.
- ***TDmonthly Classic Toy Award*** in 2007.
- ***TDmonthly Top Seller*** in 2010.
- ***NAPPA Children's Products Honors Winner*** in 2010.
- ***Sharp As A Tack OEValue*** Cognitive Processing Speed in 2013
- ***ASTRA Best Toy for Kids Award*** Classic Toy Finalist in 2013.

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